

# Weighted node-disjoint $S$ -paths

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## Goal:

Common generalization of

- Mader's Theorem on node-disjoint  $S$ -paths
  - Edmonds' maximum matching
  - Edmonds' maximum weight matching for certain weights
  - Menger's disjoint  $s$ - $t$  paths
- Min cost max  $s$ - $t$  flow
- Some results from previous talk of Hiroshi Hirai

## Outline:

- Sketch of Mader's Theorem
- Extended LP for node-disjoint  $S$ -paths
- Integrality of LP
- Good characterization
- Combinatorial min-max
- Questions

# Maximum Weight Node-Disjoint $S$ -Paths Problem

Given  $G = (V, E)$  and  $S \subseteq V$  as above.

Also given weights:

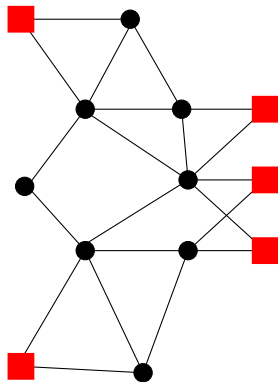
$$w : \binom{S}{2} \rightarrow \mathbb{R}_+$$

## Maximum Weight disjoint $S$ -paths problem

Given  $G, S, w$ , find

$$\max \sum_i w(sp_i, tp_i),$$

where  $P_1, P_2, \dots$  are node-disjoint  $S$ -paths.



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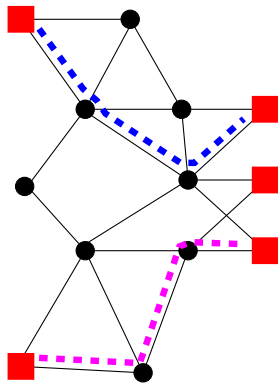
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## Same as egde-disjoint/edge-capacitated case:

Classification of weight functions

Truncated tree-metric weights:

- Maximum weight node-disjoint  $S$ -paths  $\in \mathcal{P}$
- Maximum weight node-capacitated multifold  $\in \mathcal{P}$
- Combinatorial min-max formula
- Dual integrality
- Strongly polynomial time algorithm
- Special LP has integer optimum

Not truncated tree metric weights:

- Maximum weight node-disjoint  $S$ -paths  $\in \mathcal{NP-hard}$
- Maximum weight node-capacitated multifold  $\in \mathcal{NP-hard}$
- Special LP may not have integer optimum

## Difference from egde-disjoint case:

Duality:

- No known relationship with T-joins
- No known relationship with other multifold dualities

LP:

- No known separation algorithm

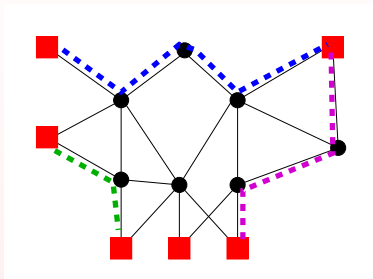
Min cost:

- No known solution with edge-cost

# Mader's Theorem on Node-Disjoint $S$ -paths

Input:

- a graph  $G = (V, E)$
- a stable set  $S \subseteq V$



## Definition

$S$ -path: a path that starts and ends in  $S$

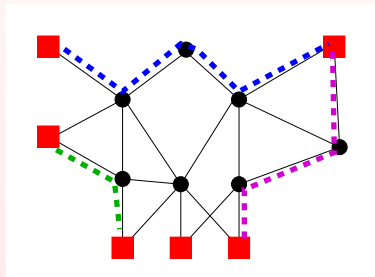
## Problem

$\nu(G, S) =$  "max number of *internally* node-disjoint  $S$ -paths"

# Mader's Theorem on Node-Disjoint $S$ -paths

## Problem

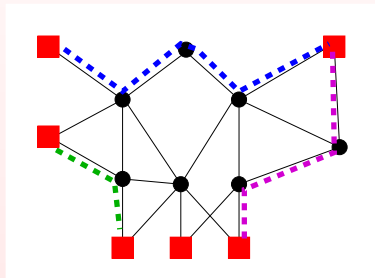
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# Mader's Theorem on Node-Disjoint $S$ -paths

## Problem

$\nu(G, S) =$  "max number of *internally* node-disjoint  $S$ -paths"



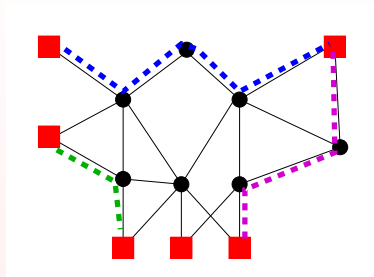
## Claim

Without loss of generality, we may assume that  $\Gamma_G(s) : s \in S$  are pairwise disjoint.

# Mader's Theorem on Node-Disjoint $S$ -paths

Input:

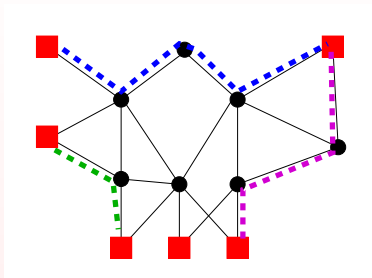
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- a set  $S \subseteq V$  such that  
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# Mader's Theorem on Node-Disjoint $S$ -paths

Input:

- a graph  $G = (V, E)$
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 $\Gamma_G(s) : s \in S$  are pairwise disjoint



Theorem (Mader's Theorem, 1978)

$$\nu(G, S) = \min_{\mathcal{X}} \left\{ |X_0| + \sum_K \left\lfloor \frac{|K \cap \cup_i X_i|}{2} \right\rfloor \right\}$$

where  $\mathcal{X} = \{X_0\} \cup \{X_s : s \in S\}$  is a partition of  $V$  such that  $\Gamma_G(s) \subseteq X_0 \cup X_s$ , and  $K$  is a component of  $G - X_0 - \cup_i E[X_i]$ .

# LP formulation for node-disjoint $S$ -paths

One idea, but not good enough:

$$x : \{S\text{-paths}\} \rightarrow \mathbb{R}_+$$

$$\max \sum_P w(s_P, t_P) x(P)$$

subject to

$$\sum_{v \in V(P)} x(P) \leq 1 \quad \text{for all } v \in V$$

$$\sum_{|V(P) \cap B| \geq 2} x(P) \leq \left\lfloor \frac{1}{2} |B| \right\rfloor \quad \text{for all } B \subseteq V - S$$

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True:

If  $w$  is a truncated tree metric, then this LP has integral optimum.

Problem with this:

This result by itself would not imply no good characterization, because the constraint matrix may have exponential rank.

# LP formulation for node-disjoint $S$ -paths

Define:

$$\overleftrightarrow{E} := \{uv : uv \in E\} \cup \{vu : uv \in E\}.$$

Introduce  $O(|S|^2|V|^3)$  variables

$$y_{ij}(uv, vz) \in \mathbb{R}_+$$

for  $i, j \in S, i \neq j, uv, vz \in \overleftrightarrow{E}, u, z \in V, v \in V - S$ .

# LP formulation for node-disjoint $S$ -paths

Define:

$$y_{ij}^{out}(uv) := \sum_{vz} y_{ij}(uv, vz)$$

$$y_{ij}^{in}(vz) := \sum_{uv} y_{ij}(uv, vz)$$

Flow Conservation Constraint:

$$y_{ij}^{out}(uv) = y_{ij}^{in}(uv) \quad \text{for } uv \in \overleftarrow{E}, u \in V - j, v \in V - i$$

# LP formulation for node-disjoint $S$ -paths

Define:

$$y^{thru}(v) := \sum_{i,j,uv,vz} y_{ij}(uv, vz).$$

Node Capacity Constraints:

$$y^{thru}(v) \leq 1 \quad \text{for } v \in V - S.$$

# LP formulation for node-disjoint $S$ -paths

Integer solutions for

Flow Conservation Constraint:

$$y_{ij}^{out}(uv) = y_{ij}^{in}(uv) \quad \text{for } uv \in \overleftrightarrow{E}, u \in V - j, v \in V - i$$

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correspond to a set of disjoint  $S$ -paths.

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Node Capacity Constraints:

$$y^{thru}(v) \leq 1 \quad \text{for } v \in V - S.$$

correspond to a set of disjoint  $S$ -paths.

Unfortunately, there may not be an integer optimum for the objective function corresponding to truncated tree metric weights.

# LP formulation for node-disjoint $S$ -paths

Call pair  $U, B$  an *odd cut* if  $|B|$  is odd,  $B \subseteq U \subseteq V - S$ .

For odd cuts  $U, B$ , define:

$$y^{via}(U, B) := \sum_{\substack{i, j \in S, b \in B \\ u \in V - U, v \in U}} (y_{ij}(ub, bv) + y_{ij}(vb, bu))$$

$$y^{jump}(U, B) := \sum_{\substack{i, j \in S, v \in U - B \\ u \in U, z \in V - U}} (y_{ij}(uv, vz) + y_{ij}(zv, vu))$$

Parity Constraints:

$$y^{jump}(U, B) \geq y^{via}(U, B) - |B| + 1 \quad \text{for odd cuts } U, B.$$

# LP formulation for node-disjoint $S$ -paths

Call triple  $U, B, F$  an *odd cut* if  $|B|$  is odd, and it satisfies  $B \subseteq U \subseteq V - S$ ,  $E[U, U - B] \subseteq F \subseteq E[U]$ .

For odd cuts  $U, B, F$ , define:

$$y^{via}(U, F, B) := \sum_{\substack{i, j \in S, b \in B \\ ub \in E - F, bv \in F}} (y_{ij}(ub, bv) + y_{ij}(vb, bu))$$

$$y^{jump}(U, F, B) := \sum_{\substack{i, j \in S, v \in U - B \\ uv \in F, vz \in E - F}} (y_{ij}(uv, vz) + y_{ij}(zv, vu))$$

Parity Constraints:

$$y^{jump}(U, F, B) \geq y^{via}(U, F, B) - |B| + 1 \quad \text{for odd cuts } U, F, B.$$

# LP for node-disjoint $S$ -paths

$$\max \sum_{i,j} w(i,j) \text{ size}(y_{ij})$$

subject to

$$y_{ij}^{\text{out}}(uv) = y_{ij}^{\text{in}}(uv) \quad \text{for } uv \in \overleftrightarrow{E}, u \in V - j, v \in V - i$$

(Flow Conservation)

$$y^{\text{thru}}(v) \leq 1 \quad \text{for } v \in V - S.$$

(Node Capacity)

$$y^{\text{jump}}(U, F, B) \geq y^{\text{via}}(U, F, B) - |B| + 1 \quad \text{for odd cuts } U, F, B$$

(Parity)

## Theorem, P, 2010

If  $w$  is a truncated tree metric, then the above LP has an integral optimum.

## Corollary

Good characterization for weighted node-disjoint  $S$ -paths, in case  $w$  is a truncated tree metric.

# LP for node-capacitated multiflows

$$\max \sum_{i,j} w(i,j) \text{ size}(y_{ij})$$

subject to

$$y_{ij}^{\text{out}}(uv) = y_{ij}^{\text{in}}(uv) \quad \text{for } uv \in \overleftrightarrow{E}, u \in V - j, v \in V - i$$

(Flow Conservation)

$$y^{\text{thru}}(v) \leq b(v) \quad \text{for } v \in V - S.$$

(Node Capacity)

$$y^{\text{jump}}(U, F, B) \geq y^{\text{via}}(U, F, B) - |B| + 1 \quad \text{for odd cuts } U, F, B$$

(Parity)

## Theorem, P, 2010

If  $w$  is a truncated tree metric, then the above LP has an integral optimum.

## Corollary

Good characterization for weighted node-capacitated multiflows, in case  $w$  is a truncated tree metric.

Consider

- $x : V \rightarrow \mathbb{R}_+$ , a set  $I$ ,  $q_i : U_i \rightarrow \overline{T}$  for  $i \in I \cup \{0\}$ , and  $y_i \geq 0$ ,  $U_i, E_i, B_i$  for  $i \in I$  such that  $U_i \subseteq V - S$ ,  $B_i \subseteq U_i$ .

Assume that

- $\{E_i : i \in I\}$  is laminar
- $E_l \cap E_{l'} = \emptyset$  implies  $(U_l - B_l) \cap (U_{l'} - B_{l'}) = \emptyset$
- $E_l \subseteq E_{l'}$  implies  $(U_l - B_l) \subseteq (U_{l'} - B_{l'})$  and  $U_l \subseteq U_{l'}$
- $E_0 = E$ ,  $U_0 = V$ , and  $q_i(v) = q_0(v)$  for all  $v \in U_i - B_i$

Define

- $i(uv) := i$  if  $E_i$  is inclusionwise minimal in  $\{E_i : i \in I \cup \{0\}\}$  with  $uv \in E_i$
- $parent(i) := j \in I \cup \{0\}$  if  $E_j$  is the inclusionwise minimal proper

The instance above is a feasible dual if it satisfies the constraints

- $q_0(s) = \phi(s)$  and  $x(s) = \varrho(s)$  for all  $s \in S$
- $2 \text{dist}(q_i(v), q_{parent(i)}(v)) \leq y_i$  for all  $v \in B_i$
- $2 \text{dist}(q_{i(uv)}(u), q_{i(uv)}(v)) \leq x(u) + x(v) - \sum y_l |\Delta(U_l - B_l, V - U_l)|$  for  $uv \in E$

The value of a feasible dual is defined by

- $value(x, I, q, y, \mathcal{E}, \mathcal{U}, \mathcal{B}) := x(V - S) + \sum_{i \in I} y_i \left\lfloor \frac{|B_i|}{2} \right\rfloor$ .

## Theorem, P, 2010

Let  $G, S, w$  be as above, and assume that  $w$  is equal to a truncated tree metric.

Then

$$\nu(G, S, w) = \min \text{value}(x, l, q, y, \mathcal{E}, \mathcal{U}, \mathcal{B})$$

where the minimum is taken over a feasible dual  $x, l, q, y, \mathcal{E}, \mathcal{U}, \mathcal{B}$ .

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where the minimum is taken over a feasible dual  $x, l, q, y, \mathcal{E}, \mathcal{U}, \mathcal{B}$ .

Implies the previous "LP integrality" result by using

- $x(v)$  for the dual variables corresponding to  $v$ , and
- $y_i$  for the dual variables corresponding to  $U_i, E_i, B_i$ .

## Theorem, P, 2010

Let  $G, S, w$  be as above, and assume that  $w$  is equal to a truncated tree metric.

Moreover, let  $b : V \rightarrow \mathbb{Z}_+$  be node-capacities.

Then

$$\nu(G, S, w) = \min \text{value}_b(x, l, q, y, \mathcal{E}, \mathcal{U}, \mathcal{B})$$

where the minimum is taken over a feasible dual  $x, l, q, y, \mathcal{E}, \mathcal{U}, \mathcal{B}$ .

$$\text{value}_b(x, l, q, y, \mathcal{E}, \mathcal{U}, \mathcal{B}) := \sum_{v \in V-S} x(v)b(v) + \sum_{i \in I} y_i \left\lfloor \frac{b(B_i)}{2} \right\rfloor$$

## Corollary, Hirai, P, 2010

Maximum weight edge-capacitated multiflow problem is tractable for truncated tree metric weights.

## Questions:

- Relationship with matroid matching?
- Is there a separation algorithm for the “odd cut” constraints?
- Closer connection with nice results on the edge-disjoint version?
- Edge cost?

Thank you for your attention!