Discrete Fuglede conjecture and Pompeiu problem

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We present a discrete version of two old problems (Fuglede and Pompeiu) originated in analysis that were investigated by many researcher and we develop a strong connection between them. This connection help us to provide new results for Fuglede's conjecture in the discrete setting.

Pompeiu raised the following question [4] . Take a continuous function f on the plane whose integral on every unit disc is zero. Does it follow that f is the constant zero function? The answer is no, but the question initiated several different type of investigations including many positive results as well.

Fuglede conjectured [1] that a bounded domain $S \subset \mathbb{R}^d$ tiles the d-dimensional Euclidean space if and only if the set of $L^2(S)$ functions admits an orthogonal basis of exponential functions.

The discrete version of Fuglede's conjecture might be formulated in the following way. A subset S of a finite abelian group G tiles G if and only if the character table of G has a submatrix, whose rows are indexed by the elements of S, which is a complex Hadamard matrix. Both directions of Fuglede's conjecture were disproved by Tao [5] and the proof is based on a counterexample for elementary abelian p-groups of finite rank. This result led to the first counterexample for the original problem in the continuous case.

In order to find answers for Pompeiu type problems one has to investigate the eigenvalues of the Cayley graph $Cay(G, S)$, that shows the connection of these two problems.

It is worth to investigate Fuglede's conjecture for finite cyclic groups since every tiling of $\mathbb{Z}$ is periodic so it originates in a tiling of a finite cyclic group. However, not much is known for cyclic groups. A recent paper of Malikiosis and Kolountzakis [3] shows that Fuglede's conjecture holds for cyclic group of order $p^n q$, where p and q are different primes.

Our main contribution towards Fuglede's conjecture for cyclic groups is to connect this problem with the Pompeiu problem, introduce more combinatorial ideas and verify it for yet unknown cases: cyclic groups of order p^2q^2 and pqr, where r is also a prime.

Further we give a neat and combinatorial proof for the previously known fact (proved by Iosevich, Mayeli and Pakianathan [2]) that Fuglede's conjecture holds for $\mathbb{Z}_p^2$.

References

[1] B. Fuglede. Commuting self-adjoint partial differential operators and a group theoretic problem. *Journal of Functional Analysis*, **16**(1), 101–121, 1974.

[2] A. Iosevich, A. Mayeli, J. Pakianathan. The Fuglede conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_p$. *Analysis & PDE*, **10**(4), 757–764, 2017.

[3] R. D. Malikiosis, M. N. Kolountzakis. Fuglede's conjecture on cyclic groups of order $p^n q$. *Discrete Analysis*, 2017:12, 16pp.

[4] D. Pompeiu. Sur une propriété intégrale des fonctions de deux variables réelles. *Bull. Sci. Acad. Roy. Belgique*, **5**(15), 265–269, 1929.

[5] T. Tao. Fuglede's conjecture is false in 5 and higher dimensions. *Mathematical Research Letters*, **11**(2), 251–258, 2004.