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György Kiss, Nicola Pace, Angelo Sonnino

ONE–FACTORIZATIONS OF THE COMPLETE GRAPH $K_{p+1}$
ARISING FROM PARABOLAS

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MANUSCRIPTS
One-factorizations of the complete graph $K_{p+1}$
arising from parabolas

György Kiss*  Nicola Pace†  Angelo Sonnino‡
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Abstract

There are three types of affine regular polygons in AG(2, q): ellipse, hyperbola and parabola. The first two cases have been investigated in previous papers. In this note, a particular class of geometric one-factorizations of the complete graph $K_n$ arising from parabolas is constructed and described in full detail. With the support of computer aided investigation, it is also conjectured that up to isomorphisms this is the only one-factorization where each one-factor is either represented by a line or a parabola.

Keywords: complete graph, one-factorization, parabola, finite plane.
Mathematics Subject Classifications: 05C70, 51E21, 05B25

1 Introduction

For a positive even integer $n$, a one-factorization of the complete graph $K_n$ is a partition of the edge set into $n - 1$ one-factors—each consisting of $\frac{n}{2}$ edges

*György Kiss: kissgy@cs.elte.hu
Department of Geometry and MTA-ELTE Geometric and Algebraic Combinatorics Research Group - Eötvös Loránd University - Pázmány s. 1/c, 1117 Budapest (Hungary), and FAMNIT - University of Primorska - Glagoljaška 8 - 6000 Koper (Slovenia)
†Nicola Pace: nicolaonline@libero.it
ITK Engineering GmbH, Herriotstr. 4, 60528 Frankfurt am Main (Germany).
‡Angelo Sonnino: angelo.sonnino@unibas.it
Dipartimento di Matematica, Informatica ed Economia - Università degli Studi della Basilicata - Viale dell’Ateneo Lucano 10 - 85100 Potenza (Italy).
partitioning the vertex set.

One-factorizations of complete graphs play a crucial role in many practical applications, like for instance scheduling tournaments, where a round robin tournament is to be played in the minimum number of sessions. Besides applications, one-factorizations have strong connections to Design Theory; see for instance [13].

Our approach to the problem of constructing one-factorizations of complete graphs is essentially geometric, as in [3, 6, 9, 10], and is based on techniques that have previously been used to find one-factorizations of multigraphs; see for instance [2, 4, 7, 11].

Basically, there are three types of affine regular polygons in the finite affine plane AG(2, q). One-factorizations arising from ellipses and hyperbolas have already been addressed in [6, 9]. In this paper the remaining case, the parabola, is investigated.

Our main result is the construction of a parabolic one-factorization—that is, a one-factorization where all one-factors except one are represented by parabolas, and the remaining one is represented by a line—for every complete graph $K_{p+1}$ with $p$ an odd prime. We may also provide a classification of parabolic one-factorizations.

Our notation is standard. For general information about one-factorizations of complete graphs see for instance [8, 12, 13].

2 Preliminaries

Henceforth we assume that $p \geq 3$ is a prime number. We fix a projective frame in PG$(2, p)$ with homogeneous coordinates $(X_0: X_1: X_2)$, and consider PG$(2, p)$ as AG$(2, p) \cup \ell_{\infty}$ where $\ell_{\infty}$ has equation $X_0 = 0$. As usual, the points of AG$(2, p)$ are written as $(X, Y)$ with $X = \frac{X_1}{X_0}$ and $Y = \frac{X_2}{X_0}$.

In AG$(2, p)$, let $\mathcal{P}_a$ be the parabola with affine equation $Y = X^2 + a$, where $a$ varies in $\mathbb{Z}_p$, and $V_\infty = (0:0:1)$ the point at infinity of the line $X_1 = 0$. Note that, in the projective closure of AG$(2, p)$, any two parabolas $\mathcal{P}_a$ and $\mathcal{P}_b$, with $a \neq b$, meet at the point $V_\infty$ only.

Let $V_i = (i, i^2)$ denote the points on $\mathcal{P}_0$ for $i = 0, 1, \ldots, p - 1$. For $k = 1, 2, \ldots, \frac{p-1}{2}$, let $P^k_i$ denote the pole of the line $V_iV_{i+k}$ with respect to $\mathcal{P}_0$. The equation of the tangent line $t_i$ to $\mathcal{P}_0$ at $V_i$ is

$$t_i : Y = 2iX - i^2.$$
hence the coordinates of the point $P^k_i = t_i \cap t_{i+k}$ are

$$P^k_i = \left( i + \frac{k}{2}, t^2 + ik \right).$$

see Figure 1. Further, let $P^\infty_i$ denote the point at infinity of the line $t_i$, that is, $P^\infty_i = (0:1:2i)$.

**Lemma 2.1.** For a fixed $k$, the points $P^k_0, P^k_1, \ldots, P^k_{p-1}$ are on the parabola $P_{\frac{k^2}{4}}$.

**Proof.** The claim follows from the equality

$$i^2 + ik = \left( i + \frac{k}{2} \right)^2 - \frac{k^2}{4}.$$

The vertices of the complete graph $K_{p+1}$ correspond to the points of $P_0 \cup \{V_\infty\}$, while the edges of $K_{p+1}$ correspond to the points of type $P^k_i$, with $k = 1, 2, \ldots, \frac{p-1}{2}, \infty$. Thus the set of edges of $K_{p+1}$ corresponds to the set of points

$$\mathcal{E} = \left( \bigcup_{k=1}^{\frac{p-1}{2}} P_{\frac{k^2}{4}} \right) \cup \left( \ell_\infty \setminus \{V_\infty\} \right).$$

These points are called external points with respect to $P_0$.

In this setting, a one-factor of $K_{p+1}$ is a set consisting of $\frac{p+1}{2}$ points of type $P^k_i$, for $i \in \{0, 1, \ldots, p-1\}$ and $k \in \{1, 2, \ldots, \frac{p-1}{2}\} \cup \{\infty\}$, satisfying the tangent property, that is, no tangent to $P_0$ meets the set in more than one point; see [6]. Then, a one-factorization of $K_{p+1}$ is just a partition of all the points of type $P^k_i$ into $p$ one-factors.

## 3 Results

Remark that a parabola of type $P_a$ cannot contain any point of type $P^\infty_j$, therefore a subset of its points satisfying the tangent property consists of at most $\frac{p-1}{2}$ points. If the line $\ell$ is not a tangent to $P_0$, then $\ell$ is called a secant if $|\ell \cap P_0| = 2$ and $\ell$ is called an external line if $|\ell \cap P_0| = 0$. It is well known (see e.g. [5, Lemma 6.14]) that a secant contains $\frac{p-1}{2}$ points of $\mathcal{E}$ and an external line contains $\frac{p+1}{2}$ points of $\mathcal{E}$. These motivate the following definitions.
Definition 3.1. A one-factor represented by a parabola $\mathcal{P}_a$ is a set of $\frac{p-1}{2}$ points of type $P_i^k$ on $\mathcal{P}_a$, together with a suitable point at infinity. A one-factor so defined is referred to as a parabolic one-factor.

Definition 3.2. A one-factor represented by a secant line $\ell$ of $\mathcal{P}_0$ is a set consisting of $\frac{p-1}{2}$ points of $\mathcal{E}$ on $\ell$, plus the pole of $\ell$ with respect to $\mathcal{P}_0$.

A one-factor represented by an external line $\ell$ of $\mathcal{P}_0$ is a set consisting of $\frac{p+1}{2}$ points of $\mathcal{E}$ on $\ell$.

Definition 3.3. A one-factorization of $K_{p+1}$ is called a parabolic one-factorization if $p - 1$ of its one-factors are represented by parabolas and one of its one-factors is represented by a line.

Theorem 3.4. Let $p$ be an odd prime. Then the complete graph $K_{p+1}$ has a parabolic one-factorization.

Proof. The proof is constructive. Let

$$F_0 = \left\{ P_{-\frac{1}{2}}^k : k = 1, 2, \ldots, \frac{p-1}{2} \right\} \cup \{ P_0^\infty \}.$$ 

The set $F_0$ is a one-factor represented by the secant line of $\mathcal{P}_0$ of equation $X = 0$, and $P_0^\infty$ is its pole with respect to $\mathcal{P}_0$. 

\[ \]
For $k = 1, 2, \ldots, \frac{p-1}{2}$, define the following sets of points:

\[ G_k = \left\{ P_{\frac{k}{2} + 2jk} : j = 0, 1, \ldots, \frac{p-3}{2} \right\} \cup \left\{ P_{\frac{-3}{2}} \right\}, \]

\[ H_k = \left\{ P_{\frac{k}{2} + (2j+1)k} : j = 0, 1, \ldots, \frac{p-3}{2} \right\} \cup \left\{ P_{\frac{3}{2}} \right\}. \]

By Lemma 2.1, $G_k \setminus \left\{ P_{\frac{-3}{2}} \right\}$ and $H_k \setminus \left\{ P_{\frac{3}{2}} \right\}$ are disjoint subsets of the parabola $\mathcal{P}_{-\frac{3}{2}}$. Both $G_k$ and $H_k$ are one-factors represented by the parabola $\mathcal{P}_{-\frac{3}{2}}$ because every tangent to $\mathcal{P}_0$ intersects $\mathcal{P}_{-\frac{3}{2}}$ in two points, $P_i^k$ and $P_i^{k+1}$. One of these points falls in $G_k$, the other one in $H_k$, and the claim follows.

Parabolic one-factorisations are completely characterised in the projective closure of $AG(2, p)$.

**Theorem 3.5.** Let $p > 5$ be an odd prime and $\mathcal{F}$ be a parabolic one-factorization of the complete graph $K_{p+1}$. Then $\mathcal{F}$ is isomorphic to the one-factorization constructed in Theorem 3.4.

**Proof.** Let $\ell$ be the line representing the unique linear one-factor of $\mathcal{F}$ and $L$ denote the pole of $\ell$ with respect to $\mathcal{P}_0$. First, we show that $\ell$ contains the point $V_\infty$. By definition, $\ell \cup \{L\}$ must contain one affine point from each parabola of type $\mathcal{P}_a$. Hence $\ell$ must be a tangent to at least $\frac{p-1}{2} - 1 > 1$ parabolas of type $\mathcal{P}_a$. Suppose that the affine equation of $\ell$ is $Y = mX + b$. Then $\ell$ contains exactly one point of $\mathcal{P}_a$ if and only if the discriminant of the quadratic equation $X^2 - mX + a - b = 0$ is zero, that is,

\[ a = \frac{m^2 + 4b}{4}. \tag{1} \]

From (1), the line $\ell$ would be a tangent to at most one parabola of type $\mathcal{P}_a$, hence it must be assumed that the affine equation of $\ell$ is of type $X = c$.

Now consider the linear transformation $\varphi \in \text{PGL}(3, p)$ associated to the matrix

\[
\begin{pmatrix}
1 & -c & c^2 \\
0 & 1 & -2c \\
0 & 0 & 1
\end{pmatrix}.
\]

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Then \((1 : c : c^2)^\varphi = (1 : 0 : 0)\) and \((0 : 0 : 1)^\varphi = (0 : 0 : 1)\). Hence, the unique linear one-factor of \(\mathcal{F}^\varphi\) corresponds, by projectivity, to the line \(X = 0\), that is, the set of points

\[
\left\{ P_{\frac{k}{2}} : k = 1, 2, \ldots, \frac{p-1}{2} \right\} \cup \{ P_0^\infty \}.
\]

Further, the linear transformation \(\varphi\) fixes every parabola \(\mathcal{P}_a\) setwise since \((1 : t : t^2 + a)^\varphi = (1 : t - c : (t - c)^2 + a)\).

For a fixed \(k \in \{1, 2, \ldots, \frac{p-1}{2}\}\) let \(G_k\) and \(H_k\) denote the two one-factors of \(\mathcal{F}^\varphi\) which are represented by the parabola \(\mathcal{P}_{\frac{k}{2}}\). Consider the point \(P_k^k\).

We may assume without loss of generality that it belongs to \(G_k\). Then, by the tangent property, \(P_k^k\) must belong to \(H_k\). For \(j = 1, \ldots, \frac{p-3}{2}\), the points \(P_{\frac{k}{2} + 2jk}^k\) must belong to \(G_k\), while the points \(P_{\frac{k}{2} + (2j+1)k}^k\) must belong to \(H_k\). Furthermore, \(P_\infty^k\) is in \(G_k\) and \(P_\infty^k\) is in \(H_k\). Thus, \(\mathcal{F}^\varphi\) is the one-factorization constructed in Theorem 3.4 and hence \(\mathcal{F}\) is isomorphic to \(\mathcal{F}^\varphi\). \(\square\)

We conclude with a conjecture that is supported by our computer aided investigations. With the aid of Magma \([1]\) we verified that the conjecture holds true for \(p \leq 17\).

**Conjecture 3.6.** Let \(p > 7\) be an odd prime, \(\mathcal{F}\) be a one-factorization of the complete graph \(K_{p+1}\) such that each one-factor of \(\mathcal{F}\) is either represented by a line or a parabola. Then \(\mathcal{F}\) is either a parabolic one-factorization or each one-factor of \(\mathcal{F}\) is represented by a line.

## 4 Examples for small \(p\)

The examples described in this section serve to illustrate the results from the previous sections.

### 4.1 \(p = 7\)

Let us consider the parabola \(\mathcal{P}_0\) of projective equation \(X_0X_2 = X_1^2\) in \(\text{PG}(2, 7)\). The construction in Theorem 3.4 provides the following partition
of the points of type $P^k_i$:

$$F_0 = \{ P^4_3(1:0:5), P^2_6(1:0:6), P^5_2(1:0:3), P^\infty_6(0:1:0) \},$$
$$F_1 = \{ P^4_4(1:1:6), P^4_6(1:3:0), P^1_4(1:5:2), P^\infty_3(0:1:6) \},$$
$$F'_1 = \{ P^5_3(1:2:2), P^3_6(1:4:0), P^2_2(1:6:6), P^\infty_4(0:1:1) \},$$
$$F_2 = \{ P^5_1(1:2:3), P^2_5(1:6:0), P^2_2(1:3:1), P^\infty_6(0:1:5) \},$$
$$F'_2 = \{ P^5_5(1:4:1), P^3_6(1:1:0), P^2_2(1:5:3), P^\infty_4(0:1:2) \},$$
$$F_3 = \{ P^3_3(1:3:5), P^3_4(1:2:0), P^3_3(1:1:4), P^2_2(0:1:4) \},$$
$$F'_3 = \{ P^3_4(1:6:4), P^2_6(1:5:0), P^2_6(1:4:5), P^\infty_4(0:1:3) \}.$$

This partition is a parabolic one-factorization, where the one-factors are as follows:

- $F_0$ is represented by the secant line $X_1 = 0$,
- $F_1, F'_1$ are represented by the parabola $P_5 : X_0X_2 = X_1^2 + 5X_0^2$,
- $F_2, F'_2$ are represented by the parabola $P_6 : X_0X_2 = X_1^2 + 6X_0^2$,
- $F_3, F'_3$ are represented by the parabola $P_3 : X_0X_2 = X_1^2 + 3X_0^2$.

### 4.2 $p = 11$

Let us consider the parabola $P_9$ of projective equation $X_0X_2 = X_1^2$ in PG(2, 11). The construction in Theorem 3.4 provides the following partition of the points of type $P^k_i$:

$$F_0 = \{ P^2_{10}(1:0:10), P^4_3(1:0:8), P^5_3(1:0:2), P^2_4(1:0:7), P^3_4(1:0:6), P^\infty_6(0:1:0) \},$$
$$F_1 = \{ P^2_3(1:1:9), P^4_3(1:3:6), P^3_{10}(1:5:0), P^4_1(1:7:2), P^3_4(1:9:1), P^\infty_7(0:1:10) \},$$
$$F'_1 = \{ P^2_4(1:2:1), P^4_9(1:4:2), P^4_6(1:6:0), P^3_2(1:8:6), P^4_4(1:10:9), P^\infty_6(0:1:1) \},$$
$$F_2 = \{ P^2_2(1:2:3), P^2_3(1:6:2), P^2_7(1:10:0), P^2_2(1:3:8), P^2_6(1:7:4), P^2_9(0:1:9) \},$$
$$F'_2 = \{ P^2_5(1:4:4), P^2_7(1:8:8), P^2_8(1:1:0), P^3_4(1:5:2), P^2_3(1:9:3), P^\infty_4(0:1:2) \},$$
$$F_3 = \{ P^3_9(1:3:4), P^3_9(1:9:10), P^3_2(1:4:0), P^3_3(1:10:7), P^3_8(1:5:9), P^3_5(0:1:8) \},$$
$$F'_3 = \{ P^3_{10}(1:6:9), P^3_5(1:1:7), P^3_6(1:7:0), P^3_6(1:2:10), P^3_4(1:8:4), P^3_7(0:1:3) \},$$
$$F_4 = \{ P^2_4(1:4:1), P^2_{10}(1:1:8), P^2_4(1:9:0), P^2_4(1:6:10), P^2_4(1:3:5), P^2_5(0:1:7) \},$$
$$F'_4 = \{ P^2_6(1:8:5), P^2_3(1:5:10), P^2_5(1:2:0), P^2_8(1:10:8), P^2_6(1:7:1), P^2_5(0:1:4) \},$$
$$F_5 = \{ P^3_3(1:5:5), P^2_2(1:4:7), P^2_9(1:3:0), P^2_3(1:2:6), P^2_4(1:1:3), P^2_3(0:1:6) \},$$
$$F'_5 = \{ P^2_5(1:10:3), P^3_3(1:9:6), P^2_6(1:8:0), P^2_{10}(1:7:7), P^2_9(1:6:5), P^2_5(0:1:5) \}.$$
This partition is a parabolic one-factorization, where the one-factors are as follows:

- $F_0$ is represented by the secant line $X_1 = 0$,
- $F_1, F'_1$ are represented by the parabola $P_8 : X_0 X_2 = X_1^2 + 8X_0^2$,
- $F_2, F'_2$ are represented by the parabola $P_{10} : X_0 X_2 = X_1^2 + 10X_0^2$,
- $F_3, F'_3$ are represented by the parabola $P_6 : X_0 X_2 = X_1^2 + 6X_0^2$,
- $F_4, F'_4$ are represented by the parabola $P_7 : X_0 X_2 = X_1^2 + 7X_0^2$,
- $F_5, F'_5$ are represented by the parabola $P_2 : X_0 X_2 = X_1^2 + 2X_0^2$.

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