A Collection of MTA–ELTE GAC manuscripts

Gabriela Araujo-Pardo, György Kiss, Christian Rubio-Montiel, Adrián Vázquez-Ávila

On line colorings of finite projective spaces

2017

MTA–ELTE Geometric and Algebraic Combinatorics Research Group

Hungarian Academy of Sciences
Eötvös University, Budapest

MANUSCRIPTS
On line colorings of finite projective spaces*

Gabriela Araujo-Pardo †       György Kiss ‡
Christian Rubio-Montiel ‡       Adrián Vázquez-Ávila §

February 22, 2017

Abstract

In this paper, we prove lower and upper bounds on the achromatic and the pseudoachromatic indices of the \(n\)-dimensional finite projective space of order \(q\).

1 Introduction

The results given in this paper are related to the well-known combinatorial problem called the Erdős-Faber-Lovász Conjecture (for short EFL Conjec-
tured), see [12].

Let \( S \) be a finite linear space. A coloring of \( S \) with \( k \) colors is an assignment of the lines of \( S \) to a set of colors \( [k] := \{1, \ldots, k\} \). A coloring of \( S \) is called proper if any two intersecting lines have different colors. The chromatic index \( \chi'(S) \) of \( S \) is the smallest \( k \) such that there exists a proper coloring of \( S \) with \( k \) colors. Erdős, Faber and Lovász conjectured ([12, 13]) that the chromatic index of any finite linear space \( S \) cannot exceed the number of its points, so if \( S \) has \( v \) points then

\[
\chi'(S) \leq v.
\]

In [8] the EFL Conjecture was proved for one of the most studied linear spaces, namely for the \( n \)-dimensional finite projective space of order \( q \), \( \text{PG}(n, q) \). In this case it is known that

\[
\chi'(\text{PG}(n, q)) \leq \frac{q^{n+1} - 1}{q - 1}.
\]

Some authors of this paper proved the EFL Conjecture for some linear spaces ([4, 5]). Moreover, in [1, 2, 3] they have been considered different types of colorations that expand the notion of the chromatic index for graphs: the achromatic and the pseudoachromatic indices. Related problems were intensively studied by several authors, see [9, 14, 15, 17]. Furthermore, in [11] Colbourn and Colbourn investigated these parameters for block designs (see also [18]).

A coloring of \( S \) is called complete if each pair of colors appears on at least one point of \( S \). It is not hard to see that any proper coloring of \( S \) with \( \chi'(S) \) colors is a complete coloring. The achromatic index \( \alpha'(S) \) of \( S \) is the largest \( k \) such that there exists a proper and complete coloring of \( S \) with \( k \) colors. The pseudoachromatic index \( \psi'(S) \) of \( S \) is the largest \( k \) such that there exists a complete coloring (not necessarily proper) of \( S \) with \( k \) colors. Clearly we have that

\[
\chi'(S) \leq \alpha'(S) \leq \psi'(S).
\]  

(1)

If \( \Pi_q \) is an arbitrary (not necessarily desarguesian) finite projective plane of order \( q \), then

\[
\chi'(\Pi_q) = \alpha'(\Pi_q) = \psi'(\Pi_q) = q^2 + q + 1,
\]
because any two lines of $\Pi_q$ have a point in common. The situation is much more complicated in higher dimensional projective spaces, the exact values of the chromatic indices are not known for $n \geq 3$. The aim of this paper is to study the achromatic and pseudoachromatic indices of finite projective spaces. Our main results are summarized in the following theorem.

**Theorem 1.1.** Let $v = \frac{q^{n+1} - 1}{q-1}$ denote the number of points of $\text{PG}(n, q)$.

1. If $n = 3 \cdot 2^i - 1$ ($i = 1, 2, \ldots$) then

   \[ \frac{1}{q} v^{\frac{1}{2}} \leq \alpha'(\text{PG}(n, q)), \]

   where $\frac{1}{2^\frac{i}{2}} \leq c_n < \frac{1}{2^\frac{i}{2}}$ is a constant which depends only on the dimension.

2. If $n \geq 2$ arbitrary integer then

   \[ \psi'(\text{PG}(n, q)) \leq \frac{1}{q} v^{\frac{1}{2}}. \]

In Section 2, we collect some known properties of projective spaces, spreads and packings, and we prove a lemma about the existence of a particular spread. In Section 3, we prove the main theorems about the achromatic and pseudoachromatic indices. Finally, in Section 4 (and Appendix is attached, where) we consider the smallest projective space, $\text{PG}(3, 2)$, and determine the exact value of its pseudoachromatic index without using a computer.

## 2 On projective spaces

It is well-known that, for any $n > 2$, the $n$-dimensional finite projective space of order $q$ exists if and only if $q$ is a prime power and it is unique up to isomorphism. Let $V_{n+1}$ be an $(n + 1)$-dimensional vector space over the Galois field $\text{GF}(q)$ of $q$ elements. The $n$-dimensional finite projective space, denoted by $\text{PG}(n, q)$, is the geometry whose $k$-dimensional subspaces for $k = 0, 1, \ldots, n$ are the $(k + 1)$-dimensional subspaces of $V_{n+1}$. For the detailed description of these spaces we refer to [16].
The basic combinatorial properties of $\text{PG}(n, q)$ can be described by the $q$-nomial coefficients $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$. This number is defined as

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})},$$

and it equals to the number of $k$-dimensional subspaces in an $n$-dimensional vector space over GF($q$). The proof of the following proposition is straightforward.

**Proposition 2.1.** In $\text{PG}(n, q)$

- the number of $k$-dimensional subspaces is $\left[ \begin{array}{c} n+1 \\ k+1 \end{array} \right]_q$, in particular, the number of points equals to $\frac{q^{n+1}-1}{q-1}$ and the number of lines equals to $\frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}$;
- the number of $k$-dimensional subspaces through a given $m$-dimensional $(m \leq k)$ subspace is $\left[ \begin{array}{c} n-m \\ k-m \end{array} \right]_q$.

A $t$-spread $\mathcal{S}^t$ of $\text{PG}(n, q)$ is a set of $t$-dimensional subspaces (for short $t$-subspaces) of $\text{PG}(n, q)$ which partitions $\text{PG}(n, q)$, that is, each point of $\text{PG}(n, q)$ lies in exactly one element of $\mathcal{S}^t$. Hence any two elements of $\mathcal{S}^t$ are disjoint. A 1-spread is also called line spread and it is denoted by $\mathcal{S}$. It is well-known that a $t$-spread of $\text{PG}(n, q)$ exists if and only if $(t+1)|(n+1)$, hence line spreads exist in projective spaces of odd dimension.

A $t$-packing $\mathcal{P}^t$ of $\text{PG}(n, q)$ is a partition of the $t$-spaces of $\text{PG}(n, q)$ into $t$-spreads. A 1-packing is also called line packing or parallelism and it is denoted by $\mathcal{P}$. The next result is an obvious corollary of Proposition 2.1.

**Proposition 2.2.**

- A $t$-spread in $\text{PG}(n, q)$ consists of $\frac{q^{n+1}-1}{q^{t+1}-1}$ $t$-subspaces.
- A $t$-packing in $\text{PG}(n, q)$ consists of $\left[ \begin{array}{c} n \\ t \end{array} \right]_q$ $t$-spreads.

A necessary and sufficient condition for the existence of a $t$-packing of $\text{PG}(n, q)$ is not known in general. The following theorems give specific constructions in some particular cases:
Theorem 2.3 (Beutelspacher [7]). If \( n = 2^i - 1 \) with \( i = 1, 2, \ldots \) then for all prime power \( q \) the finite projective space \( \text{PG}(2^i - 1, q) \) admits a line packing.

Theorem 2.4 (Baker [6]). For all integer \( m > 0 \) the finite projective space \( \text{PG}(2m + 1, 2) \) admits a line packing.

A regulus of \( \text{PG}(3, q) \) is a set \( \mathcal{R} \) of \( q + 1 \) mutually skew lines such that any line of \( \text{PG}(3, q) \) intersecting three distinct elements of \( \mathcal{R} \) intersects all elements of \( \mathcal{R} \). It is known [10] that any three pairwise skew lines \( \ell_1, \ell_2, \ell_3 \) of \( \text{PG}(3, q) \) are contained in exactly one regulus \( \mathcal{R} = \mathcal{R}(\ell_1, \ell_2, \ell_3) \) of \( \text{PG}(3, q) \). A line spread \( \mathcal{I} \) of \( \text{PG}(3, q) \) is called regular, if for any three distinct lines of \( \mathcal{I} \) the whole regulus \( \mathcal{R} = \mathcal{R}(\ell_1, \ell_2, \ell_3) \) is contained in \( \mathcal{I} \).

Theorem 2.5 (Beutelspacher [7]). For any regular spread \( \mathcal{I} \) of \( \text{PG}(3, q) \) there is a packing \( \mathcal{P} \) of \( \text{PG}(3, q) \) which contains \( \mathcal{I} \) as one of its spreads.

There is an important class of spreads. The notion of geometric spread was introduced by Segre [19] in the following way. Let \( \langle X, Y \rangle \) be the subspace of \( \text{PG}(n, q) \) generated by \( X \) and \( Y \), where \( X \) and \( Y \) are two different elements of a \( t \)-spread \( \mathcal{I}^t \) of \( \text{PG}(n, q) \). As \( X \) and \( Y \) are disjoint, from the dimension formula we get that \( \langle X, Y \rangle \) is a \( (2t+1) \)-subspace. We say that \( \mathcal{I}^t \) induces a spread \( \mathcal{I}^t_{\langle X, Y \rangle} \) in \( \langle X, Y \rangle \), if any element \( Z \) of \( \mathcal{I}^t \) having at least one point in \( \langle X, Y \rangle \) is totally contained in \( \langle X, Y \rangle \). The \( t \)-spread \( \mathcal{I}^t \) is called geometric if \( \mathcal{I}^t \) induces a spread \( \mathcal{I}^t_{\langle X, Y \rangle} \) in \( \langle X, Y \rangle \) for any two distinct elements \( X \) and \( Y \) of \( \mathcal{I}^t \).

It is not difficult to check (see [7], Section 4) that a \( t \)-spread \( \mathcal{I}^t \) of \( \text{PG}(n, q) \) is geometric if and only if the following holds. If the elements \( X \) of \( \mathcal{I}^t \) are called large points, and the subspaces \( \langle X, Y \rangle \) are called large lines, then the large points and large lines form a projective space. This space, \( \Pi_{\mathcal{I}^t} \), has dimension \( s = \frac{n+1}{t+1} - 1 \) and order \( q^{t+1} \), it is isomorphic to \( \text{PG}\left(\frac{n+1}{t+1} - 1, q^{t+1}\right) \).

The following two results due to Segre [19].

Theorem 2.6. The finite projective space \( \text{PG}(n, q) \) admits a geometric \( t \)-spread if and only if there exists a positive integer \( s \) such that \( n + 1 = (t + 1)(s + 1) \) holds.
Lemma 2.7. If $\text{PG}(n,q)$ admits a geometric line spread $\mathcal{S}$ then $\mathcal{S}_{X,Y}$ is a regular line spread of the 3-dimensional subspace $(X,Y)$ of $\text{PG}(n,q)$ for any $X,Y \in \mathcal{S}$ ($X \neq Y$).

Combining the cited results of Beutelspacher and Segre, we prove a lemma that plays a crucial role in the proof of the lower bound in Theorem 1.1.

If $n = 3 \cdot 2^i - 1$ ($i = 1, 2, \ldots$) then $n + 1 = (2^i - 1 + 1)(2 + 1)$, hence the projective space $\text{PG}(n,q)$ admits a geometric $t$-spread $\mathcal{S}^t$ with $t = 2^i - 1$. The large points and large lines form a projective plane $\Pi_{\mathcal{S}^t}$ of order $q^{t+1}$.

Consider the lines of $\Pi_{\mathcal{S}^t}$ and denote the corresponding $(2^{t+1} - 1)$-subspaces of $\text{PG}(n,q)$ by $\mathcal{L}_j$ ($j = 1, \ldots, q^{2t+2} + q^{t+1} + 1$). The $t$-spread $\mathcal{S}^t$ is geometric, therefore for all $j$ the elements $X$ of $\mathcal{S}^t$ with $X \cap \mathcal{L}_j \neq \emptyset$ form a $t$-spread of $\mathcal{L}_j$ which will be denoted by $\mathcal{S}^t_j$. The spread $\mathcal{S}^t_j$ induces a special line packing of $\mathcal{L}_j$.

Lemma 2.8. Let $\text{PG}(n,q)$ be the finite projective space of dimension $n = 3 \cdot 2^i - 1$ ($i = 1, 2, \ldots$). Then there exists a geometric $t$-spread $\mathcal{S}^t$ with $t = 2^i - 1$ having the property that any finite projective subspace $\mathcal{L}_j$ admits a line packing $\mathcal{P}_j$ such that the set of lines contained in $\mathcal{S}^t_j$ is the union of elements of some line spreads of $\mathcal{P}_j$.

Proof. Since $n+1 = (1+1)((3 \cdot 2^{i-1} - 1) + 1)$, it follows from Theorem 2.6 that $\text{PG}(3 \cdot 2^i - 1, q)$ admits a geometric line spread $\mathcal{S}$. The elements of $\mathcal{S}$ and the 3-subspaces $(X,Y) \in \mathcal{S}$, $X \in \mathcal{S}$, $Y \neq X$) can be considered, respectively, as points and lines of a $(3 \cdot 2^{i-1} - 1)$-dimensional space $\Pi_\mathcal{S}$ of order $q^2$. Denote the 3-subspaces of $\text{PG}(3 \cdot 2^i - 1, q)$ corresponding to the lines of $\Pi_\mathcal{S}$ by $\mathcal{U}_j$, where $j = 1, \ldots, [3^{2^{i-1}} - 1]q^2$. Since $\mathcal{S}$ is a geometric spread, as a consequence of Lemma 2.7, we have that the elements $X$ of $\mathcal{S}$ with $X \cap \mathcal{U}_j \neq \emptyset$ form a regular line spread of $\mathcal{U}_j$ which will be denoted by $\mathcal{S}_{\mathcal{U}_j}$. Moreover, by Theorem 2.5, we conclude that the 3-space $\mathcal{U}_j$ admits a packing $\mathcal{P}_j$ such that $\mathcal{S}_{\mathcal{U}_j} \in \mathcal{P}_j$. For $k = 1, 2, \ldots, q^2 + q$ let $\mathcal{S}_{j,k}$ be the other spreads of $\mathcal{P}_j$, hence

$$\mathcal{P}_j = \{\mathcal{S}_{\mathcal{U}_j}, \mathcal{S}_{j,1}, \ldots, \mathcal{S}_{j,q^2+q}\}.$$
We claim that the set
\[ \mathcal{P} = \bigcup_{j=1}^{q^4+q^2+1} (\mathcal{P}_j \setminus \{\mathcal{L}_j\}) \cup \mathcal{I} \]
is equal to the line set of $\text{PG}(n, q)$. The lines of $\mathcal{I}$ obviously appears in $\mathcal{P}$ exactly once. If a line $\ell \notin \mathcal{I}$, then $\ell$ lies in a unique subspace of type $\mathcal{L}_j$. Namely, if the lines $e, f, g, h \in \mathcal{I}$ meet $\ell$ then $\ell \subset \langle e, f \rangle$ and $\ell \subset \langle g, h \rangle$, but this means that $g \cap \langle e, f \rangle \neq \emptyset$ and $h \cap \langle e, f \rangle \neq \emptyset$. Since $\mathcal{I}$ is geometric this implies that $g$ and $h$ are contained in $\langle e, f \rangle$ and therefore $\langle e, f \rangle = \langle g, h \rangle$. But $\mathcal{P}$ contains exactly one packing of $\mathcal{L}_j$, hence each line of $\text{PG}(n, q)$ appears in $\mathcal{P}$ exactly once.

Now, we prove the statement of the theorem by induction on $i$.

If $i = 1$ then it follows from Theorem 2.6 that $\text{PG}(5, q)$ admits a geometric line spread $\mathcal{I}$. The elements of $\mathcal{I}$ and the 3-spaces $\langle X, Y \mid X, Y \in \mathcal{I}, X \neq Y \rangle$ can be considered as points and lines of a plane $\Pi_{\mathcal{I}}$ of order $q^2$, respectively. Denote the 3-spaces of $\text{PG}(5, q)$ corresponding to the lines of $\Pi_{\mathcal{I}}$ by $\mathcal{L}_j$, where $j = 1, \ldots, q^4 + q^2 + 1$. Since $\mathcal{I}$ is a geometric spread, Lemma 2.7 gives that the elements $X$ of $\mathcal{I}$ with $X \cap \mathcal{L}_j \neq \emptyset$ form a regular line spread of $\mathcal{L}_j$ which will be denoted by $\mathcal{I}_{\mathcal{L}_j}$. Because of Theorem 2.5 the 3-space $\mathcal{L}_j$ admits a packing $\mathcal{P}_j$ such that $\mathcal{I}_{\mathcal{L}_j} \subset \mathcal{P}_j$. For $i = 1, 2, \ldots, q^2 + q$ let $\mathcal{P}_{j,1}$ be the other spreads of $\mathcal{P}_j$, hence
\[ \mathcal{P}_j = \{\mathcal{I}_{\mathcal{L}_j}, \mathcal{I}_{j,1}, \ldots, \mathcal{I}_{j,q^2+q}\}. \]

Consider now the case $i > 1$ and let us assume that the assertion of Lemma 2.8 is proved for all $i' < i$. Since $n + 1 = 3 \cdot 2^i = (1 + 1)(3 \cdot 2^{i-1} - 1 + 1)$, by Theorem 2.6, $\text{PG}(n, q)$ admits a geometric 1-spread $\mathcal{I}$. As before, we consider the elements $X$ of $\mathcal{I}$ and the 3-subspaces $\langle X, Y \mid X, Y \in \mathcal{I}, X \neq Y \rangle$ as points and lines of a $(3 \cdot 2^{i-1} - 1)$-space $\Pi_\mathcal{I}$ of order $q^2$, respectively. Denote the lines of $\Pi_\mathcal{I}$ by $\mathcal{B}_k$, where $k = 1, \ldots, M$ where $M$ is the number of lines of $\Pi_\mathcal{I}$. The spread $\mathcal{I}$ is geometric, therefore the elements $X$ of $\mathcal{I}$ with $X \cap \mathcal{B}_k \neq \emptyset$ form a spread of $\mathcal{B}_k$ which will be denoted by $\mathcal{I}_k$. By Lemma 2.7, $\mathcal{I}_k$ is a regular spread of $\mathcal{B}_k$. According to Theorem 2.5, in all $\mathcal{B}_k$ there exists a packing $\mathcal{P}_{\mathcal{B}_k}$ of $\mathcal{B}_k$ which contains $\mathcal{I}_k$ as one
of its spreads. Let this packing be
\[ \mathcal{P}_k = \{ \mathcal{I}_{k,0}, \ldots, \mathcal{I}_{k,q^2+q} \}. \]
with \( \mathcal{I}_{k,0} = \mathcal{I}_k \).

Hence by induction, \( \Pi_{\mathcal{I}} \) admits a basic construction \( \mathcal{C}_i \) with the property that any finite projective subspaces \( \mathcal{U}_j \) admits a packing \( \mathcal{P}_j \) such that the lines contained in \( \mathcal{I}_j \) are the union of elements of \( \mathcal{P}_j \). Let \( \mathcal{I}_j^t = \{ \mathcal{I}_{j,1}, \ldots, \mathcal{I}_{j,u} \} \) then \( \mathcal{P}_j = \mathcal{I}_j^t \cup \{ \mathcal{I}_{j,u+1}, \ldots, \mathcal{I}_{j,v} \} \) where \( v \) is the number of 1-spreads in \( \mathcal{U}_j \) of \( \Pi_{\mathcal{I}} \).

Recall that each line of \( \Pi_{\mathcal{I}} \) is a 3-subspace of PG\((n,q)\). If \( \mathcal{I}_{j,l} = \{ u_{l(1)}, \ldots, u_{l(w)} : 1 \leq l \leq v \} \) where \( w \) is the number of lines in a 1-spreads of \( \mathcal{U}_j \) (as a subspace of \( \Pi_{\mathcal{I}} \)), then \( \mathcal{I}_{j,l,m} = \{ \mathcal{I}_{l(1)}, \ldots, \mathcal{I}_{l(w)} : 0 \leq m \leq q^2 + q \} \) is a 1-spread of \( \mathcal{U}_j \) (as a subspace of PG\((n,q)\)).

We construct the following packing \( \mathcal{P}_j \) of \( \mathcal{U}_j \) (like subspace of PG\((n,q)\)):
\[ \mathcal{P}_j = \bigcup_{l=1}^{u} q^2 + q \mathcal{I}_{j,l,m} \cup \bigcup_{l=1}^{u} q^2 + q \mathcal{I}_{j,l,m}. \]
By construction, in \( \bigcup_{l=1}^{u} q^2 + q \mathcal{I}_{j,l,m} \) are all the lines of \( \mathcal{I}_j^t \) and the lemma follows.

\[ \square \]

3 On line colorings of projective spaces

First, we introduce some notions that we use to prove our results. Let \( \mathcal{L} \) be the set of lines of PG\((n,q)\) and \( \mathcal{P} \) its set of points. Given a coloring \( \zeta : \mathcal{L} \to [k] \) with \( k \) colors, we say that a point \( p \in \mathcal{P} \) is an owner of a set of colors \( C \subseteq [k] \) whenever for every \( c \in C \) there is \( q \in \mathcal{P} \), such that \( \zeta((p,q)) = c \); and given a line \( l \) of \( \mathcal{L} \) we say that \( l \) is an owner of a set of colors \( C \subseteq [k] \), if each point of \( l \) is an owner of \( C \). Therefore, we say that \( \zeta \) is a complete coloring if for every pair of colors in \( [k] \) there is a point in \( \mathcal{P} \) which is an owner of both colors.

**Lemma 3.1.** Suppose that \( n+1 = (t+1)(2+1) \) and let \( \mathcal{I}^t \) be a geometric \( t \)-spread of PG\((n,q)\). Suppose that each large line \( \mathcal{U}_j \) of \( \Pi_{\mathcal{I}^t} \) is an owner of
a set of colors \( C_j \subseteq C \). Then for every pair of colors \( \{c_1, c_2\} \subseteq \bigcup_j C_i \) there is a point \( x \) of \( \text{PG}(n, q) \), which is an owner of \( c_1 \) and \( c_2 \).

Proof. Let \( N \) denote the number of large lines of \( \Pi_{\mathcal{J}^t} \) and let \( \{c_1, c_2\} \subseteq \bigcup_i C_i \) be an arbitrary subset. Since \( \mathcal{U}_i \) is the owner of \( C_i \), if there exists \( i \in \{1, \ldots, N\} \) such that \( \{c_1, c_2\} \subseteq C_i \), then it follows that each \( x \in \mathcal{U}_i \) is an owner of \( c_1 \) and \( c_2 \). If \( c_1 \in C_i \) and \( c_2 \in C_j \) with \( i \neq j \), then there exists a point \( x \in \mathcal{J}_i \), such that \( x \) is a point of the \( t \)-space \( \mathcal{U}_i \cap \mathcal{U}_j \), and since \( \mathcal{U}_i \) and \( \mathcal{U}_j \) are owners of \( C_i \) and \( C_j \), respectively, this implies that \( x \) is an owner of \( c_1 \) and \( c_2 \). \qed

3.1 Lower bound

Now, we are ready to prove the lower bound of Theorem 1.1.

Proof of Theorem 1.1, Part 1. By Lemma 2.8, \( \text{PG}(n, q) \) admits a basic construction \( C_i \) with the property that any finite projective subspace \( \mathcal{U}_j \) admits a packing \( \mathcal{P}_j \) such that the set of lines contained in \( \mathcal{J}_j \) is the union of elements of \( \mathcal{P}_j \).

Let \( \mathcal{J}_j^t = \{ \mathcal{J}_{j,1}, \ldots, \mathcal{J}_{j,r}: r = [n/\mathcal{J}_j]\} \), where \( r \) is the number of 1-spreads of \( \mathcal{J}_j \), then \( \mathcal{J}_j = \mathcal{J}_j^t \cup \{ \mathcal{J}_{j,r+1}, \ldots, \mathcal{J}_{j,s}: s = [n+1/\mathcal{J}_j]\} \) where \( s \) is the number of 1-spreads of \( \mathcal{J}_j \). Note that the number of 1-spreads in \( \mathcal{J}_j^s := \mathcal{J}_j \setminus \mathcal{J}_j^t = \{ \mathcal{J}_{j,r+1}, \ldots, \mathcal{J}_{j,s}\} \) is \( s - r = q^{[n+1]/\mathcal{J}_j} \).

We color every line in a fixed \( \mathcal{J}_{j,k} \) with the color \( j(k-r) \) for \( r+1 \leq k \leq s \). Every element \( X \) of \( \mathcal{J}_j^t \) is a \( t \)-subspace, by Theorem 2.3, admits a packing \( \mathcal{P}_X = \{ \mathcal{J}_{1,X}, \ldots, \mathcal{J}_{[n/\mathcal{J}_j],X}\} \). We color every line in a fixed \( \mathcal{J}_{i,X} \) with the color \( (s-r)N + i \).

Observe that the coloring is proper and, as a consequence of Lemma 3.1, it is also complete. In the coloring we use

\[
(s-r)N + r = q^t \frac{q^{n+1} - 1}{q-1} + \frac{q^t - 1}{q-1} = \frac{q^{n+t+1} - 1}{q-1}
\]

colors. Let \( h = \frac{4n+1}{3n} \). Since \( n + t + 1 = \frac{4n+1}{3} = hn \) and \( 2q^n > \frac{q^{n+1} - 1}{q-1} = v \),

9
we have\
\[\frac{q^{n+t+1} - 1}{q - 1} = \frac{q^{hn} - 1}{q - 1} > \frac{1}{2^{\frac{h}{q}}} \cdot \frac{(2q^n)^h}{q} > \frac{1}{2^{\frac{h}{q}}} \cdot \frac{v^h}{q},\]

hence Inequality 2 holds with \( c_n = \frac{1}{2^r} \), and the theorem follows, because 5 ≤ n implies \( \frac{7}{5} \leq h < \frac{4}{3} \).

3.2 Upper bound

Now, we prove the upper bound for the pseudoachromatic index of the smallest finite projective spaces PG(n, q).

Proof of Theorem 1.1, Part 2. If \( r \) denotes the number of lines through a fixed point, then the total number of point-line incidences is \( v^{\binom{r}{2}} \). Hence \( v^{\binom{r}{2}} \geq (\psi'(PG(n,q))) \). Solving this quadratic inequality we get

\[\psi'(PG(n,q)) \leq \frac{1 + \sqrt{1 + 4vr(r-1)}}{2}.
\]

Since \( \sqrt{1 + 4vr(r-1)} \leq \sqrt{4vr^2} - 1 \) and \( r = \frac{v-1}{q} \), this gives

\[\psi'(PG(n,q)) \leq \sqrt{vr} = \frac{1}{q} \sqrt{v(v-1)}
\]

and the result follows.

4 The case of PG(3, 2)

In this section, we determine the pseudoachromatic index of the smallest finite projective space, PG(3, 2), in a pure combinatorial way, without using any computer aided calculations. To do this, we need some lemmas about pencils and null polarities.

Definition 4.1. Let \( \Pi \) be a plane and \( P \in \Pi \) be a point in PG(3, q). A pencil with carrier \( P \) in \( \Pi \) is the set of the \( q+1 \) lines of PG(3, q) through \( P \) which are contained in \( \Pi \).

Lemma 4.2. Let \( \mathcal{E} \) be a set of five lines in PG(3, 2). If any two lines of \( \mathcal{E} \) have a point in common then \( \mathcal{E} \) contains a pencil.
Definition 4.3. Let PG(3, q)' denote the dual space of PG(3, q), and let A be a 4 × 4 non-singular matrix over GF(q) satisfying the equation $A = -A^T$.

A null polarity $\pi : PG(3, q) \to PG(3, q)'$ is a collineation which maps the point with coordinate vector $x$ to the point with coordinate vector $xA$.

As the points, lines and planes of the dual space are planes, lines and points of the original space, respectively, a null polarity maps lines of PG(3, q) to lines of PG(3, q). A null polarity maps intersecting lines to intersecting lines, hence the proof of the following statement is straightforward.

Lemma 4.4. Let $\pi$ be a null polarity and $\varsigma$ be a line-coloring of PG(3, q). Then $\varsigma$ is complete if and only if $\varsigma \circ \pi^{-1}$ is a complete line-coloring of PG(3, q)'.

Theorem 4.5. The pseudoachromatic index of PG(3, 2) is equal to 18, i.e.,

$$\psi'(PG(3, 2)) = 18.$$
there are at least 3 color classes of size one. If there were at least five color classes of size one in $\varsigma$ then, by Lemma 4.2, we could choose three color classes such that the corresponding lines would form a pencil. Suppose that three lines, $\ell_1, \ell_2$ and $\ell_3$ form a pencil with carrier $P$ in the plane $\Pi$, and each of these lines forms a color class of size one. Consider the other 16 classes. At most 4 of them contain lines through $P$ and at most 4 of them contain lines in $\Pi$. Each of the remaining at least 8 classes must have size at least 3, because they have to meet each $\ell_i$ for $i = 1, 2, 3$. This implies that these color classes contain altogether $8 \times 3 = 24$ or more lines. As the total number of lines is 35, this means that each of the remaining 11 color classes contains exactly one line. Hence each of the seven lines through $P$, and each of the 4 lines in $\Pi$ not through $P$ are color classes of size one, but they do not meet, so $\varsigma$ is not complete. This contradiction proves the statement.

Choose three color classes of size one and let $\ell_1, \ell_2$ and $\ell_3$ be the lines in these color classes. Any two of these lines have a point in common, but they do not form a pencil, hence either they form a triangle, or they have a point in common but they are not coplanar. In the latter case apply Lemma 4.4. If the three lines meet in the point $P$ then after a null polarity $\pi$, the lines $\ell_1^\pi, \ell_2^\pi$ and $\ell_3^\pi$ form a triangle in the plane $P^\pi$. As PG(3, 2) is isomorphic to its dual space, it is enough to consider the first case.

From now on, we suppose that $\ell_1, \ell_2$ and $\ell_3$ form a triangle $ABC$ in the plane $\Pi$. Let $A', B'$ and $C'$ be the third points of the sides of the triangle, respectively, and let $D = AA' \cap BB' \cap CC'$ be the seventh point of the plane $\Pi$. Take $\Pi$ as the plane at infinity and consider the remaining eight points as AG(3, 2). The coordinates of the points in $\Pi$ can be chosen as follow. $A = (0 : 1 : 0 : 0), B = (0 : 0 : 1 : 0), C = (0 : 0 : 0 : 1), A' = (0 : 0 : 1 : 1), B' = (0 : 1 : 0 : 1), C' = (0 : 1 : 1 : 0)$ and $D = (0 : 1 : 1 : 1)$.

First, suppose that there is a 4th color class of size one and let $\ell_4$ denote the line in this class. Then $\ell_4$ must be in $\Pi$. If it contains one of the points $A, B$ or $C$, then a pencil appears, hence the coloring is not complete. So we may assume that $\ell_4$ is the line $A'B'C'$. Among the other 15 color classes there are 14 classes of size 2 and one of size 3. Consider the four lines, say $\ell_5, \ell_6, \ell_7$ and $\ell_8$, through $D$ but not in $\Pi$. If two or three of them formed a
color class, then this class would have empty intersection with each of $\ell_1, \ell_2$ and $\ell_3$, contradiction. So these four lines are distributed among at least three color classes and each class of size two must contain a line of $\Pi$. Thus there are two possibilities for these color classes.

(a) $\{\ell_5, \ell_8, AA', \ell_6, BB', \ell_7, CC'\}$,

(b) $\{\ell_5, AA', \ell_6, BB', \ell_7, CC', \ell_8, \ell_9, \ell_{10}\}$, where $\ell_9$ is a line through $A$ and $\ell_{10}$ is a line through $A'$.

Each of the remaining classes contains two lines whose points at infinity cover $\ell_1, \ell_2, \ell_3$ and $\ell_4$. Since no three of these lines have a point in common, it can happen if and only if two points at infinity of these color classes coincide with one of the sets $\{A, A'\}, \{B, B'\}$ and $\{C, C'\}$.

If there is no more color class of size one, then each of the remaining 16 classes has size 2. The pairs of the four lines through $D$ must be the four lines of $\Pi$ distinct from $\ell_1, \ell_2, \ell_3$. If an affine line passes on $A'$, then its pair must pass on $A$, and the same is true for the lines through $\{B, B'\}$ and $\{C, C'\}$.

We can summarize these possibilities as follow.

- Each of the lines $\ell_1, \ell_2$ and $\ell_3$ forms a class of size one.
- There are 12 classes such that two points at infinity of these color classes coincide with one of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}$.
- Each of the pairs $\{\ell_5, AA'\}, \{\ell_6, BB'\}$ and $\{\ell_7, CC'\}$ belong to one color class.
- The nineteenth color class contains the line $A'B'C'$.
- The line $\ell_8$ is “free”.

We can choose the system of reference such that the pair of $AA'$ is the line $DOE$ where $O = (1 : 0 : 0 : 0)$ and $E = (1 : 1 : 1 : 1)$. Let $X = (1 : 1 : 0 : 0)$, $Y = (1 : 0 : 1 : 0)$, $Z = (1 : 0 : 0 : 1)$, $K = (1 : 1 : 1 : 0)$, $L = (1 : 1 : 0 : 1)$ and $M = (1 : 0 : 1 : 1)$ be the other affine points of $\text{PG}(3, 2)$, see Figure 1.
The pair of the line $CXL$ is either the line $C'O$ or $C' E$. As the roles of $O$ and $E$ were symmetric previously, we may assume without loss of generality that $C'OK$ is the pair of $CXL$.

![Diagram](image)

Figure 1: PG(3, 2) but not all the lines are drawn.

First, consider the three other classes whose two points in $\Pi$ are $C$ and $C'$. The affine part of the three lines through $C$ are $OZ$, $MY$, $KE$, while the affine part of the three lines through $C'$ are $EZ$, $XY$, $LM$. Each of these classes must meet the line $DOE$. Hence, we need a matching between these two line-triples such that each pair contains at least one of the points $O$ and $E$. So the pair of $MY$ must be $EZ$. There are two possibilities for the remaining two pairs, so the four possible pairs through $C$ and $C'$:

i) $(XL, OK)$, $(MY, EZ)$, $(OZ, XY)$, $(KE, LM)$,

ii) $(XL, OK)$, $(MY, EZ)$, $(KE, XY)$, $(OZ, LM)$.

In Case i) take the four classes whose two points in $\Pi$ are $B$ and $B'$. The affine parts of the four lines through $B$ are $OY$, $EL$, $XK$, $ZM$, while the affine parts of the four lines through $B'$ are $OL$, $EY$, $XZ$, $MK$. Again,
we need a matching such that each pair contains at least one of the points $O$ and $E$, and each class must meet the four classes belonging to $\{C, C'\}$. So the pair of $XK$ is $EY$, because $(XK, OL)$ has empty intersection with $(MY, EZ)$. Hence the pair of $ZM$ is $OL$. The pair of $MK$ is $OY$, because $(EL, MK)$ has empty intersection with $(OZ, XY)$. So the affine parts of the four pairs belonging to $\{B, B'\}$ are 

$$(XK, EY), (ZM, OL), (OY, MK), (EL, XZ).$$

Take the four classes whose two points in $\Pi$ are $A$ and $A'$. The affine parts of the four lines through $A$ are $OX, EM, YK, ZL$, while the affine part of the four lines through $A'$ are $OM, EX, YZ, LK$. At least three classes consist of only two lines. Let us look for these classes. None of the pairs $(OX, LK), (OX, YZ), (EM, YZ), (EM, LK)$, is good, because its intersection is empty with $(YM, EZ), (KE, LM), (XL, OK), (OZ, XY)$, respectively. In the same way none of the pairs $(KY, OM), (KY, EX), (LZ, OM), (LZ, EX)$ is good because their intersection are empty with $(EL, XZ), (ZM, LO), (XK, YE), (OY, MK)$, respectively. This means that in the matching there are only four possible pairs containing $OX$ or $EM$, 

$$(OX, OM), (OX, EX), (EM, OM), (EM, EX), \quad (3)$$

and four possible pairs containing $KY$ or $LZ$, 

$$(KY, LK), (KY, YZ), (LZ, YZ), (LZ, LK). \quad (4)$$

Thus at least one pair from (3) and at least one pair from (4) form a color class.

Now consider the affine part of the two color classes containing $\{B, B'\}$ and $\{C, C'\}$. These are the lines through $D$, except $DOE$. So they consist of the points $X$ and $M$, $Z$ and $K$, $L$ and $Y$. At least one of the two classes contains only one pair of points. But the pair $\{X, M\}$ has empty intersection with any class from (4), while both pairs $\{Z, K\}$ and $\{L, Y\}$ have empty intersection with any class from (3). Hence the coloring cannot be complete in Case i).
Now consider Case ii). Take the four classes whose two points in Π are \(B, B'\). The affine parts of the four lines through \(B\) are \(OY, EL, XK, ZM\), while the affine parts of the four lines through \(B'\) are \(OL, EY, XZ, MK\). Again we need a matching such that each pair contains at least one of the points \(O\) and \(E\), and each class must meet the four classes belonging to \(\{C, C'\}\).

So the pair of \(XK\) is \(OL\), because \((XK, EY)\) has empty intersection with \((OZ, LM)\). Hence the pair of \(ZM\) is \(EY\). We distinguish two cases, according to the pair of \(OY\). So the affine parts of the four pairs belonging to \(\{B, B'\}\) are

(a) \((XK, LO), (ZM, EY), (OY, XZ), (EL, MK)\),

(b) \((XK, LO), (ZM, EY), (OY, MK), (EL, XZ)\).

Take the four classes whose two points at infinity are \(A\) and \(A'\). The affine parts of the four lines through \(A\) are \(OX, EM, YK, ZL\), while the affine part of the four lines through \(A'\) are \(OM, EX, YZ, LK\). At least three classes consist of only two lines. Let us look for these classes. In both cases none of the pairs \((OX, LK), (KY, EX), (EM, YZ), (LZ, OM)\) is good, because it has empty intersection with \((YM, EZ), (OZ, LM), (XK, LO), (KZ, XY)\), respectively.

Furthermore, in Case (a) none of the pairs \((OX, YZ), (EM, LK)\) is good, because its intersection is empty with \(\{O, X, Y, Z\} \cap \{E, L, M, K\} = \{E, M, L, K\} \cap \{O, Y, X, Z\} = \emptyset\). Thus we get four possible pairs containing \(OX\) or \(EM\):

\[
(OX, OM), (OX, EX), (EM, OM), (EM, EX), \quad (5)
\]

and six possible pairs containing \(KY\) or \(LZ\):

\[
(KY, OM), (LZ, EX),

(KY, LK), (KY, YZ), (LZ, LK), (LZ, YZ). \quad (6)
\]

If either \((KY, OM)\) or \((LZ, EX)\) belongs to the matching, then only one more pair from (5) can be in it, hence at least one more pair from (6) also
belongs to the matching. Thus at least one pair from (5) and at least one pair from (6) form a color class.

In Case (b) none of the pairs \((LZ, EX), (KY, OM)\) is good, because it has empty intersection with \((OY, MK), (EL, XZ)\), respectively. Thus we get six possible pairs containing \(OX\) or \(EM\):

\[
(OX, YZ), (EM, LK),
\]

\[
(OX, OM), (OX, EX), (EM, OM), (EM, EX),
\]

and four possible pairs containing \(KY\) or \(LZ\):

\[
(KY, LK), (KY, YZ), (LZ, LK), (LZ, YZ).
\]

If either \((OX, YZ)\) or \((EM, LX)\) belongs to the matching, then only one more pair from (7) can be in it, hence at least one more pair from (8) also belongs to the matching. Thus at least one pair from (7) and at least one pair from (8) form a color class.

Finally, in both Cases (a) and (b), consider the affine part of the two color classes containing \(\{B, B'\}\) and \(\{C, C'\}\). These are the lines through \(D\), except \(DOE\). So they consist of the points \(X\) and \(M\), \(Z\) and \(K\), \(L\) and \(Y\). At least one of the two classes contains only one pair of points. But the pair \(\{X, M\}\) has empty intersection with any class from (6) and from(8), while both of the pairs \(\{Z, K\}\) and \(\{L, Y\}\) have empty intersection with any class from (5) and from (7). Hence the coloring cannot be complete in Case ii).

Now, we present a complete coloring with 18 color classes.

Let the lines \(\ell_1, \ell_2, \ell_3\) and \(\ell_4 = A'B'C'\) form color classes of size one. These classes are denoted by \(C_1, C_2, C_3\) and \(C_4\), respectively. The class \(C_5\) consists of the lines \(AA'D\) and \(OED\), while the class \(C_6\) consists of the remaining five lines through \(D\). Any two of these six classes obviously have non-empty intersection. The remaining twelve classes of size two are formed by the \(3 \times 4\) pairs of lines whose points at infinity are \(\{A, A'\}, \{B, B'\}\) and \(\{C, C'\}\), respectively. The affine parts of these classes are the following:

\[
C_{A1}: (OX, EX), \quad C_{A2}: (OM, EM), \quad C_{A3}: (YK, YZ), \quad C_{A4}: (LZ, LK);
\]
$C_{B1}$: $(OY, MX)$, $C_{B2}$: $(XK, EY)$, $C_{B3}$: $(ZM, OL)$, $C_{B4}$: $(EL, XZ)$; $C_{C1}$: $(OZ, XY)$, $C_{C2}$: $(XL, OK)$, $C_{C3}$: $(YM, EZ)$, $C_{C4}$: $(KE, LM)$.

If $1 \leq i \leq 6$ then $C_i$ contains at least one element of each of the pairs \{A, A'\}, \{B, B'\} and \{C, C'\}, and any two color classes belonging to the same quadruple of classes of type $C_{Qi}$ also intersect each other. Hence it is enough to show that $C_{Qi}$ and $C_{Rj}$ have non-empty intersection if $Q \neq R$. The following three tables give one point of intersection in each case.

<table>
<thead>
<tr>
<th></th>
<th>$C_{B1}$</th>
<th>$C_{B2}$</th>
<th>$C_{B3}$</th>
<th>$C_{B4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{A1}$</td>
<td>O</td>
<td>X</td>
<td>O</td>
<td>X</td>
</tr>
<tr>
<td>$C_{A2}$</td>
<td>M</td>
<td>E</td>
<td>M</td>
<td>E</td>
</tr>
<tr>
<td>$C_{A3}$</td>
<td>Y</td>
<td>Y</td>
<td>Z</td>
<td>Z</td>
</tr>
<tr>
<td>$C_{A4}$</td>
<td>K</td>
<td>K</td>
<td>L</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$C_{C1}$</th>
<th>$C_{C2}$</th>
<th>$C_{C3}$</th>
<th>$C_{C4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{A1}$</td>
<td>O</td>
<td>O</td>
<td>E</td>
<td>E</td>
</tr>
<tr>
<td>$C_{A2}$</td>
<td>O</td>
<td>O</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>$C_{A3}$</td>
<td>Y</td>
<td>K</td>
<td>Y</td>
<td>K</td>
</tr>
<tr>
<td>$C_{A4}$</td>
<td>Z</td>
<td>L</td>
<td>Z</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$C_{C1}$</th>
<th>$C_{C2}$</th>
<th>$C_{C3}$</th>
<th>$C_{C4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{B1}$</td>
<td>O</td>
<td>O</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>$C_{B2}$</td>
<td>X</td>
<td>X</td>
<td>E</td>
<td>E</td>
</tr>
<tr>
<td>$C_{B3}$</td>
<td>O</td>
<td>O</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>$C_{B4}$</td>
<td>X</td>
<td>X</td>
<td>E</td>
<td>E</td>
</tr>
</tbody>
</table>

This proves that the coloring is complete. \qed

References


