A Collection of ELKH–ELTE GAC manuscripts

Bence Csajbók, Péter Sziklai, Zsuzsa Weiner

Renitent lines

2021

ELKH
Eötvös Loránd Research Network

ELKH–ELTE Geometric and Algebraic Combinatorics Research Group

successor of the MTA-ELTE GAC Research group of the Hungarian Academy of Sciences and the Eötvös University, Budapest

MANUSCRIPTS
Renitent lines
Bence Csajbók, Peter Sziklai, Zsuzsa Weiner*

Abstract
There are many examples for point sets in finite geometry, which behave “almost regularly” in some (well-defined) sense, for instance they have “almost regular” line-intersection numbers. In this paper we investigate point sets of a desarguesian affine plane, for which there exist some (sometimes: many) parallel classes of lines, such that almost all lines of one parallel class intersect our set in the same number of points (possibly mod $p$, the characteristic). The lines with exceptional intersection numbers are called renitent, and we prove results on the (regular) behaviour of these renitent lines.

1 Introduction
One of the key motivations in the history of finite geometries is the study of symmetric structures, i.e. structures admitting a large symmetry group. These structures (quadrics, Hermitian varieties, subgeometries over a subfield, etc.) are typically very “regular” when you consider their intersection properties with the subspaces of the ambient geometry; and there exist many “classification-type” results, stating that an “intersection-wise very regular” set must be one on the list of the (classical, symmetric) structures.

A natural next step is to investigate point sets, which behave “almost regularly” with respect to the subspaces of the ambient space. In this paper we restrict ourselves to point sets of a desarguesian affine plane $\text{AG}(2,q)$, where $q$ is a power of the prime $p$. (The natural but not obvious extensions to other spaces will come in separate papers.) It may well happen, that our point set intersects almost all lines of a parallel class in the same number of points (possibly mod $p$). If it happens for many parallel classes then one may guess that the reason is that our point set has a hidden structure, i.e. the non-regular intersections may be “corrected”, or at least they also possess some regularity themselves. For instance, in $\text{AG}(2,q)$, $q$ even, an arc of size $q+1$ has almost regular intersection numbers mod 2 in every parallel line class: it meets each line in 0 or 2 points, except one line per each parallel class (which is 1-secant). Now, it is easy to prove that these 1-secant (exceptional) lines are concurrent, which can be interpreted as (i) they are points of a dual curve of degree 1; or (ii) their intersection point can be added to the original point set, which becomes “regular” this way (i.e. every line meets it in 0 mod 2 points). There are many papers dealing with problems in view of (ii), for a survey on arcs see [1] and for $k \pmod{p}$ multisets see [10]. In this paper we extend and explore this idea. We start with a result, which is from [5] and it can be viewed as a generalization of [3, Theorem 5], see also [2, Proposition 2] and [9, Remark 7].

*The first author was supported by the ÚNKP-19-4 New National Excellence Program of the Ministry for Innovation and Technology and by the National Research, Development and Innovation Office – NKFIH, grant no. PD 132463. The first and the third author acknowledges the partial support of the National Research, Development and Innovation Office – NKFIH, grant no. K 124950. The second author is grateful for the partial support of project K 120154 of the National Research, Development and Innovation Fund of Hungary.
Lemma 1.1 (Lemma of renitent lines [5]). Let \( \mathcal{T} \) be a point set of \( \AG(2, q) \). A line \( \ell \) with slope \( d \) is called renitent if there exists an integer \( m_\ell \) such that \( |\ell \cap \mathcal{T}| \not\equiv m_\ell \pmod{p} \) but every other line with slope \( d \) meet \( \mathcal{T} \) in \( m_\ell \) modulo \( p \) points. The renitent lines are concurrent.

Now we define renitent lines in the following, more general setting and then prove various generalizations of the lemma above.

Definition 1.2. Let \( \mathcal{M} \) be a multiset of \( \AG(2, q) \). For some integer \( \lambda \leq (q-1)/2 \) a direction \( (d) \) is called \( (q-\lambda) \)-uniform if there are at least \( (q-\lambda) \) affine lines with slope \( d \) meeting \( \mathcal{M} \) in the same number of points modulo \( p \). This number will be called the typical intersection number at \( (d) \). The rest of the lines with direction \( (d) \) will be called renitent.

A sharply \((q-\lambda)\)-uniform direction \( (d) \) is a direction incident with exactly \((q-\lambda)\) affine lines meeting \( \mathcal{M} \) in the same number of points modulo \( p \).

In the definition above different directions might have different typical intersection numbers. Note that these numbers are uniquely determined for each \((q-\lambda)\)-uniform direction because of the assumption \( \lambda \leq (q-1)/2 \).

Under some conditions, we are able to prove that the renitent lines are contained in an algebraic envelope of relatively small class (a curve of the dual plane of relatively small degree), as it is in Theorem 2.1 below.

Theorem 2.1 Take a multiset \( \mathcal{T} \) of \( \AG(2, q) \) and let \( E_\lambda \) denote a set of at most \( q \) directions which are \((q-\lambda)\)-uniform such that the following hold:

(i) \( 0 < \lambda \leq \min\{q-2, p-1\} \),
(ii) for each \((d) \in E_\lambda \) the renitent lines meet \( \mathcal{T} \) in the same number, say \( t_d \), of points modulo \( p \),
(iii) for each \((d) \in E_\lambda \) if \( m_d \) denote the typical intersection number at direction \( (d) \), then \( t_d - m_d \) modulo \( p \) does not depend on the choice of \( (d) \).

Then the renitent lines with direction in \( E_\lambda \) are contained in an algebraic envelope of class \( \lambda \).

In Theorem 2.2, we remove the condition on the size of \( E_\lambda \) and give a more general formulation of Theorem 2.1. The proof relies on \( k \)-th power sums and the Newton–Girard formulas. If we are more permissive with the renitent lines of \( \mathcal{T} \) then the class of the algebraic envelope might increase. To prove the following result we apply a “weighted” version of the Newton–Girard formulas, see Lemma 3.1.

Theorem 3.3 Take any \( \mathcal{T} \subseteq \AG(2, q) \) and an integer \( 0 < \lambda \leq (q-1)/2 \). Let \( E_\lambda \) denote a set of \((q-\lambda)\)-uniform directions of size at most \( q \). The renitent lines with slope in \( E_\lambda \) are contained in an algebraic envelope of class \( \lambda^2 \). Furthermore, if a direction is \((q-\lambda)\)-uniform, but not sharply uniform, then the line pencil centered at that direction is fully contained in the envelope.

At Section 4, we show how to apply the resultant method, cf. [10], with the help of a polynomial which can detect renitent lines at each uniform direction.

2 Envelopes of small class

In this section our aim is to show that there is an envelope of relatively small class containing the renitent lines with slope in a given subset \( \mathcal{F}_\lambda \) of \((q-\lambda)\)-uniform directions. There exist at least \( \lambda \) common lines of such an envelope and a line pencil centered at a sharply \((q-\lambda)\)-uniform direction.
Of course, the union of the line pencils centered at the \((q - \lambda)\)-uniform directions is an envelope containing these renitent lines. So the minimal class of such an envelope is at least \(\min\{|\mathcal{E}_\lambda|, \lambda\}\).

Any line set of size at most \(|\mathcal{F}_\lambda|\lambda\) is contained in an algebraic envelope of class at most \(\lfloor \sqrt{2}|\mathcal{F}_\lambda|\lambda \rfloor - 1\). Indeed, a 3-variable homogeneous polynomial \(h\) of degree at most \(\frac{d+2}{2}\) has \((\frac{d+2}{2})\) coefficients. For any point \(P \in \text{PG}(2,q)\), the condition that \(h\) vanishes at \(P\) is equivalent to a linear equation for these coefficients. If \((\frac{d+2}{2})\) is larger than the size of a point set \(S\), then there is a non-trivial solution for the homogeneous system of equations which corresponds to the condition that the points of \(S\) are zeroes of \(h\). In Theorem 2.1, we show that no matter of the size of \(\mathcal{F}_\lambda\), with some conditions on the renitent lines we can always construct an envelope of class \(\lambda\) containing all renitent lines with slope in \(\mathcal{F}_\lambda\). The results of this section will rely on the Newton–Girard formulas.

From now on, we coordinatize the projective plane \(\text{PG}(2,q)\) with homogeneous coordinates over \(\text{GF}(q)\) in such a way that the line at infinity is \([0 : 0 : 1]\), while the \(X\)- and the \(Y\)-axes are \([0 : 1 : 0]\) and \([1 : 0 : 0]\), resp. The direction \((d)\), where \(d \in \mathbb{F}_q\), means the point \((1 : d : 0)\) on the line at infinity. The slope of the affine lines incident with \((d)\) is \(d\).

Later in this section, we will be able to remove the condition \(|\mathcal{E}_\lambda| \leq q\) from Theorem 2.1, see Remark 2.3.

**Theorem 2.1.** Take a multiset \(\mathcal{T}\) of \(\text{AG}(2,q)\) and let \(\mathcal{E}_\lambda\) denote a non-empty set of at most \(q\) directions which are \((q - \lambda)\)-uniform such that the following hold:

(i) \(0 < \lambda \leq \min\{q - 2, p - 1\}\),

(ii) for each \((d)\) in \(\mathcal{E}_\lambda\) the renitent lines meet \(\mathcal{T}\) in the same number, say \(t_d\), of points modulo \(p\),

(iii) for each \((d)\) in \(\mathcal{E}_\lambda\) if \(m_d\) denote the typical intersection number at direction \((d)\), then \(t_d - m_d\) modulo \(p\) does not depend on the choice of \((d)\).

Then the renitent lines with direction in \(\mathcal{E}_\lambda\) are contained in an algebraic envelope of class \(\lambda\).

**Proof.** First we show that the number of renitent lines is the same at each direction \((d)\) of \(\mathcal{E}_\lambda\). Let \((d)\) and \((e)\) denote two directions in \(\mathcal{E}_\lambda\) which are sharply \((q - \lambda_d)\)-uniform and sharply \((q - \lambda_e)\)-uniform, respectively. Then

\[(q - \lambda_d)m_d + \lambda_dt_d \equiv |\mathcal{T}| \equiv (q - \lambda_e)m_e + \lambda_et_e \pmod{p},
\]

hence

\[\lambda_d(t_d - m_d) \equiv \lambda_e(t_e - m_e) \pmod{p}.\]

By assumption \((iii)\), \(t_d - m_d \equiv t_e - m_e \pmod{p}\) and \(t_d \neq m_d \pmod{p}\), thus \(\lambda_d \equiv \lambda_e \pmod{p}\). Then \(\lambda_d = \lambda_e\) follows from the fact that \(0 \leq \lambda_e, \lambda_d \leq \lambda \leq p - 1\).

From now on, we may assume that the directions in \(\mathcal{E}_\lambda\) are sharply \((q - \lambda)\)-uniform. Since \(|\mathcal{E}_\lambda| \leq q\), we may assume \((0 : 1 : 0) \notin \mathcal{E}_\lambda\). For each \((1 : d : 0) \in \mathcal{E}_\lambda\) put \((0 : \alpha_1(d) : 1),(0 : \alpha_2(d) : 1),\ldots,(0 : \alpha_\lambda(d) : 1)\) for the points of the \(Y\)-axis on the renitent lines with slope \(d\).

Put \(s := |\mathcal{T}|\) and \(\mathcal{T} = \{(a_i : b_i : 1)\}_{i=1}^s\). Next define the polynomials

\[\pi_k(V) := \sum_{i=1}^s (b_i - a_i)V^k \in \mathbb{F}_q[V]\]

of degree at most \(k\). The line joining \((1 : d : 0)\) and \((a_i : b_i : 1)\) meets the \(Y\)-axis at the point \((0 : b_i - a_id : 1)\), hence for each \((1 : d : 0) \in \mathcal{E}_\lambda\) the multiset

\[M_d := \{(b_i - a_id)\}_{i=1}^s\]
contains \(m_d\) modulo \(p\) copies of \(\mathbb{F}_q\) and \(c\) modulo \(p\) further copies of \(\alpha_i(d)\) for \(1 \leq i \leq \lambda\), where \(c \in \{1, \ldots, p-1\}\) is an integer such that \(c \equiv t_d - m_d \pmod{p}\). Since \(\sum_{\gamma \in \mathbb{F}_q} \gamma^k = 0\) for \(0 \leq k \leq q-2\) and since \(\pi_k(d)\) is the \(k\)-th power sum of \(M_d\), for \(0 \leq k \leq q-2\) it holds that for \((1 : d : 0) \in \mathcal{E}_\lambda\)

\[
\pi_k(d) = c \sum_{i=1}^{\lambda} \alpha_i(d)^k. \tag{1}
\]

Denote by \(\sigma_i(X_1, \ldots, X_\lambda)\) the \(i\)-th elementary symmetric polynomial in the variables \(X_1, \ldots, X_\lambda\). Also, for any integer \(i \geq 0\) and \(d \in \mathcal{E}_\lambda\) put

\[
\sigma_i(d) = \sigma_i(\alpha_1(d), \ldots, \alpha_\lambda(d)).
\]

For \(p - 1 \geq j \geq 1\) define the following polynomial of degree at most \(j\):

\[
M_j(V) := (-1)^j \sum_{n_1 + 2n_2 + \ldots + n_j = j} \prod_{i=1}^{j} \left( -\pi_i(V)/c \right)^{n_i} \in \mathbb{F}_q[V].
\]

Then for \(\min\{q - 2, p - 1\} \geq j \geq 1\) from (1) and from the Newton-Girard identities it follows that \(M_j(d) = \sigma_j(d)\) for each \((1 : d : 0) \in \mathcal{E}_\lambda\).

Consider the affine curve of degree \(\lambda\) defined by

\[
f(U, V) := U^\lambda - M_1(V)U^{\lambda - 1} + M_2(V)U^{\lambda - 2} - \ldots + (-1)^{\lambda - 1}M_{\lambda - 1}(V)U + (-1)^{\lambda}M_{\lambda}(V).
\]

Then the projective curve of degree \(\lambda\) defined by the equation \(g(U, V, W) := W^\lambda f(U/W, V/W)\) contains the point \((\alpha_i(d) : d : 1)\) for each \(d \in \mathcal{E}_\lambda\) and \(1 \leq i \leq \lambda\). Indeed,

\[
g(U, d, 1) = U^\lambda - \sigma_1(d)U^{\lambda - 1} + \sigma_2(d)U^{\lambda - 2} - \ldots + (-1)^{\lambda - 1}\sigma_{\lambda - 1}(d)U + (-1)^{\lambda}\sigma_\lambda(d) = \prod_{i=1}^{\lambda}(U - \alpha_i(d)).
\]

It follows that the lines \([d : -1 : \alpha_i(d)]\) are contained in an algebraic envelope of class \(\lambda\).

\[\Box\]

**Theorem 2.2.** Take a multiset \(\mathcal{T}\) of \(AG(2, q)\) and let \(\mathcal{F}_\lambda\) denote a set of \((q - \lambda)\)-uniform directions. For each \((d) \in \mathcal{F}_\lambda\) denote the typical intersection number by \(m_d\) and denote the intersection numbers of the renitent lines by \(t_{d,1}, t_{d,2}, \ldots, t_{d,\lambda_d}\), for some \(0 < \lambda_d \leq \lambda\). For \(c \in \mathbb{F}_p \setminus \{0\}\) define the integers \(\lambda_{d,i}(c) \in \{1, \ldots, p - 1\}\) such that

\[
c\lambda_{d,i}(c) \equiv t_{d,i} - m_d \pmod{p}
\]

and assume that

\[
\Lambda_d(c) := \sum_{i=1}^{\lambda_d} \lambda_{d,i}(c) \leq \min\{q - 2, p - 1\} \tag{2}
\]

holds for each \((d) \in \mathcal{F}_\lambda\) (note that the sum is taken over natural numbers). Then \(\Lambda(c) := \Lambda_d(c)\) does not depend on \(d\) and the renitent lines with direction in \(\mathcal{F}_\lambda\) are contained in an algebraic envelope of class \(\Lambda(c)\). Moreover, the intersection multiplicity of this envelope with the pencil centered at \((d)\) at a renitent line incident with \((d)\) and with intersection number \(t_{d,i}\) is \(\lambda_{d,i}(c)\).
Proof. Throughout the proof, we will fix $c$ and so neglect it from $\lambda_{d,i}(c)$, $\Lambda_d(c)$ and $\Lambda(c)$. As before, first we show that $\Lambda_d$ does not depend on $(d)$ of $F_\lambda$. Let $(d)$ and $(e)$ denote two directions in $F_\lambda$ and define $\Lambda_d$ and $\Lambda_e$ as in (2). Then

$$qm_d + \Lambda_d c \equiv |T| \equiv qm_e + \Lambda_e c \quad (\text{mod } p),$$

hence

$$\Lambda_d \equiv \Lambda_e \quad (\text{mod } p).$$

Then $\Lambda_d = \Lambda_e$ follows from the fact that $0 < \Lambda_e, \Lambda_d \leq p - 1$.

First we prove the assertion for any $E \subseteq F_\lambda$ such that $|E_\lambda| \leq q$. In this case, then we may assume $(0 : 1 : 0) \notin E_\lambda$. For each $(1 : d : 0) \in E_\lambda$ put $(0 : \alpha_1(d) : 1), (0 : \alpha_2(d) : 1), \ldots, (0 : \alpha_{\lambda_d}(d) : 1)$ for the points of the $Y$-axis on the renitent lines with slope $d$.

Put $s := |T|$ and $T = \{(a_i : b_i : 1)\}_{i=1}^s$. Next define the polynomials

$$\pi_k(V) := \sum_{i=1}^s (b_i - a_i V)^k \in \mathbb{F}_q[V]$$

of degree at most $k$. For each $(1 : d : 0) \in E_\lambda$ the multiset

$$M_d := \{(b_i - a_i t_d)\}_{i=1}^s$$

contains $m_d$ modulo $p$ copies of $\mathbb{F}_q$ and $t_{d,i} - m_d$ modulo $p$ further copies of $\alpha_i(d)$ for $1 \leq i \leq \lambda_d$.

Since $\sum_{\gamma \in \mathbb{F}_q} \gamma^k = 0$ for $0 \leq k \leq q - 2$ and since $\pi_k(d)$ is the $k$-th power sum of $M_d$, for $0 \leq k \leq q - 2$ it holds that for $(1 : d : 0) \in E_\lambda$

$$\pi_k(d) = c \sum_{i=1}^{\lambda_{d,i}} \sum_{j=1}^{\lambda_{d,j}} \alpha_i(d)^j.$$ (3)

Denote by $\sigma_i(X_1, \ldots, X_A)$ the $i$-th elementary symmetric polynomial in the variables $X_1, \ldots, X_A$. Also, for any integer $i \geq 0$ and $d \in E_\lambda$ put

$$\sigma_i(d) = \sigma_i(\alpha_1(d), \ldots, \alpha_i(d), \ldots, \alpha_j(d), \ldots, \alpha_{\lambda_d}(d), \ldots, \alpha_{\lambda_j}(d)).$$

For $p - 1 \geq j \geq 1$ define the following polynomial of degree at most $j$:

$$M_j(V) := (-1)^j \sum_{n_1 + 2n_2 + \ldots + jn_j = j} \prod_{i=1}^j \left(-\pi_i(V)/c\right)^{n_i} n_1! n_2! \cdots n_j! \in \mathbb{F}_q[V].$$

Then for $\min\{q - 2, p - 1\} \geq j \geq 1$ from (3) and from the Newton–Girard identities it follows that $M_j(d) = \sigma_j(d)$ for each $(1 : d : 0) \in E_\lambda$.

Consider the affine curve of degree $\Lambda$ defined by

$$f(U, V) := U^\Lambda - M_1(V)U^{\Lambda - 1} + M_2(V)U^{\Lambda - 2} - \ldots + (-1)^{\Lambda - 1} M_{\Lambda - 1}(V)U + (-1)^{\Lambda} M_\Lambda(V).$$

Then the projective curve of degree $\Lambda$ defined by the equation $g(U, V, W) := W^\Lambda f(U/W, V/W)$ contains the point $(\alpha_i(d) : d : 1)$ for each $d \in E_\lambda$ and $1 \leq i \leq \lambda_d$, with multiplicity $\lambda_{d,i}$. Indeed,

$$g(U, d, 1) = U^\Lambda - \sigma_1(d)U^{\Lambda - 1} + \sigma_2(d)U^{\Lambda - 2} - \ldots + (-1)^{\Lambda - 1} \sigma_{\Lambda - 1}(d)U + (-1)^{\Lambda} \sigma_\Lambda(d) =$$
\[ \prod_{i=1}^{\lambda_d} (U - \alpha_i(d))^{\lambda_{d,i}}. \]

It follows that the point \((\alpha_i(d) : d : 1)\) lies on the curve defined by \(f\). Note that \((V - d)\) cannot divide \(f(U, V)\) and hence the “horizontal” line \([0 : 1 : -d]\) cannot be a component of the curve defined by \(f\). The intersection multiplicity of \(f\) and \([0 : 1 : -d]\) is \(\lambda_{d,i}\) at the point \((\alpha_i(d) : d : 1)\). It follows that the lines \([d : 1 : -\alpha_i(d)]\) are contained in an algebraic envelope of class \(\Lambda\) and the intersection multiplicity of this envelope with the pencil centered at \((1 : d : 0)\) is \(\lambda_{d,i}\) at the line \([d : 1 : -\alpha_i(d)]\).

Now assume that \(|\mathcal{F}_\Lambda| = q + 1\). Apply the argument above for two distinct subsets of \(\mathcal{F}_\Lambda\), both of them of size \(q\). Denote them by \(\mathcal{E}_\Lambda\) and \(\mathcal{E}'_\Lambda\) and denote the corresponding dual curves of degree \(\Lambda\) by \(\mathcal{C}\) and \(\mathcal{C}'\), respectively. Our aim is to prove that these two curves coincide. Put \(\mathcal{C} = \mathcal{H} \cdot \mathcal{A}\) and \(\mathcal{C}' = \mathcal{H} \cdot \mathcal{A}'\), where \(\mathcal{A}\) and \(\mathcal{A}'\) do not have a common component. Denote the degree of \(\mathcal{H}\) by \(h\) and suppose for the contrary that \(h < \Lambda\).

For \((d) \in \mathcal{E}_\Lambda \cap \mathcal{E}'_\Lambda\) denote the line \([0 : 1 : -d]\) by \(\ell_d\) and the point \((\alpha_i(d) : d : 1)\) by \(P_{d,i}\). Recall that \(\ell_d\) cannot be a component of \(\mathcal{C}\) or \(\mathcal{C}'\). The intersection multiplicity \(I(\mathcal{C} \cap \ell_d, P_{d,i}) = \lambda_{d,i}\) equals \(I(\mathcal{H} \cap \ell_d, P_{d,i}) + I(\mathcal{A} \cap \ell_d, P_{d,i})\), thus

\[ I(\mathcal{A} \cap \ell_d, P_{d,i}) = \lambda_{d,i} - I(\mathcal{H} \cap \ell_d, P_{d,i}), \]

and by similar reasons

\[ I(\mathcal{A}' \cap \ell_d, P_{d,i}) = \lambda_{d,i} - I(\mathcal{H} \cap \ell_d, P_{d,i}). \]

By [7, Lemma 10.4] it follows that \(I(\mathcal{A} \cap \mathcal{A}', P_{d,i}) \geq \lambda_{d,i} - I(\mathcal{H} \cap \ell_d, P_{d,i})\). Then

\[ \sum_{i=1}^{\lambda_d} I(\mathcal{A} \cap \mathcal{A}', P_{d,i}) \geq \sum_{i=1}^{\lambda_d} \lambda_{d,i} - \sum_{i=1}^{\lambda_d} I(\mathcal{H} \cap \ell_d, P_{d,i}) = \Lambda - \sum_{i=1}^{\lambda_d} I(\mathcal{H} \cap \ell_d, P_{d,i}) \geq \Lambda - h, \]  

(4)

since \(\sum_{i=1}^{\lambda_d} I(\mathcal{H} \cap \ell_d, P_{d,i}) \leq \deg \mathcal{H} = h\). The inequality (4) holds for each \((d) \in \mathcal{E}_\Lambda \cap \mathcal{E}'_\Lambda\) and hence

\[ \sum_{P \in \mathcal{A} \cap \mathcal{A}'} I(\mathcal{A} \cap \mathcal{A}', P) \geq \sum_{(d) \in \mathcal{E}_\Lambda \cap \mathcal{E}'_\Lambda} \sum_{i=1}^{\lambda_d} I(\mathcal{A} \cap \mathcal{A}', P_{d,i}) \geq (q - 1)(\Lambda - h). \]

On the other hand, By Bézout’s theorem \(\sum_{P \in \mathcal{A} \cap \mathcal{A}'} I(\mathcal{A} \cap \mathcal{A}', P) \leq \deg \mathcal{A} \cdot \deg \mathcal{A}' = (\Lambda - h)(\Lambda - h)\).

This contradiction proves \(h = \Lambda\), that is, \(\mathcal{C} = \mathcal{C}'\).

\[ \square \]

**Remark 2.3.** If in Theorem 2.2 we assume \(t_{d,1} = t_{d,2} = \ldots = t_{d,\lambda} =: t_d\) for each \((d) \in \mathcal{F}_\Lambda\), we also assume that \(t_d - m_d\) does not depend on the choice of \(d\), and further assume \(0 < \lambda \leq \min\{q-2, p-1\}\), then with the choice \(c \equiv t_d - m_d \pmod{p}\) we obtain Theorem 2.2 without the restriction \(|\mathcal{E}_\Lambda| \leq q\).

**Remark 2.4.** If \(|T| \equiv 0 \pmod{p}\) then Theorem 2.2 cannot be applied. Indeed, in that case for each \((d) \in \mathcal{F}_\Lambda\),

\[ \sum_{i=1}^{\lambda_d} (t_{d,i} - m_d) \equiv 0 \pmod{p} \]

and hence \(\Lambda_d = \sum_{i=1}^{\lambda_d} \lambda_{d,i} \equiv 0 \pmod{p}\), which is not possible if \(\Lambda_d \leq p - 1\) (by definition \(\lambda_{d,i} > 0\) for each \(i\)).
Remark 2.5. For $c = 1, 2, \ldots, p - 1$, the values of $\Lambda_d(c)$ give different residues modulo $p$ and this residue is the same for every choice of $d$. However, it is possible that for some of the directions, $\Lambda_d(c) \leq \min\{q - 2, p - 2\}$ does not hold.

Indeed, put for example $p = 5$, $\lambda = 2$ and assume that the typical intersection number is 0 at each direction. Also, assume that both of the two renitent lines with direction $(d_1) \in \mathcal{F}_\lambda$ meet $\mathcal{T}$ in 1 modulo $p$ points and the two renitent lines with direction $(d_2) \in \mathcal{F}_\lambda$ meet $\mathcal{T}$ in 3 and in 4 points modulo $p$. Then, with $c = 1$ we obtain $\Lambda_{d_1}(c) = 2$ and $\Lambda_{d_2}(c) = 7$, so the result cannot be applied. On the other hand, with $c = 3$ we obtain $\Lambda_{d_1}(c) = \Lambda_{d_2}(c) = 4$ and hence the result can be applied if $q > 5$. It shows that sometimes it might be convenient to choose $\mathcal{F}_\lambda$ not as the set of all $(q - \lambda)$-renitent lines, but as a subset of them.

Example 2.6. In Theorem 2.2 put $\lambda = 3$ and assume that $m_d = 1$ and the renitent lines meet $\mathcal{T}$ modulo $p$ in the multiset $\{3, 3, 5\}$ for each $(d) \in \mathcal{F}_\lambda$. Note that this implies $p \neq 2$. With the choice $c = 2$ it follows that the renitent lines are contained in a curve of degree $\Lambda = (3 - 1)/2 + (3 - 1)/2 + (5 - 1)/2 = 4$ whenever $4 \leq \min\{q - 2, p - 1\}$.

One might think that to obtain a curve of the lowest degree, the best option is to chose $c$ as the greatest common divisor of the values $t_{d_1} - m_d$. The next example refute this belief.

Example 2.7. In Theorem 2.2, put $\lambda = 2$, $p = 13$, and assume that $m_d = 1$ and the renitent lines meet $\mathcal{T}$ modulo 13 in the multiset $\{2, 8\}$ for each $(d) \in \mathcal{F}_\lambda$. With the choice $c = 1$ it follows that the renitent lines are contained in a curve of degree $\Lambda = (2 - 1) + (8 - 1) = 8$. With the choice $c = 7$ it follows that the renitent lines are contained in a curve of degree $\Lambda = 1/7 + 7/7 = 2 + 1 = 3$.

3 The general case

First we prove a recursion connecting elementary symmetric polynomials with “weighted” power sums.

Lemma 3.1. Let $\sigma_k = \sigma_k(X_1, \ldots, X_\lambda)$ denote the $k$-th elementary symmetric polynomial in the variables $X_1, \ldots, X_\lambda$. For some field elements $c_1, c_2, \ldots, c_\lambda$ put

$$P_k = P_k(X_1, \ldots, X_\lambda) = \sum_{i=1}^{\lambda} c_i X_i^k.$$

Then for any integer $j \geq 0$ it holds that

$$P_{\lambda+j} = P_{\lambda+j-1} \sigma_1 - P_{\lambda+j-2} \sigma_2 + \ldots + (-1)^{\lambda+j+1} P_j \sigma_\lambda.$$

Proof. Note that

$$Y^j \prod_{i=1}^{\lambda} (Y - X_i) = Y^{\lambda+j} - Y^{\lambda+j-1} \sigma_1 + Y^{\lambda+j-2} \sigma_2 - \ldots + (-1)^\lambda Y^j \sigma_\lambda.$$

It follows that

$$c_1(X_1^{\lambda+j} - X_1^{\lambda+j-1} \sigma_1 + X_1^{\lambda+j-2} \sigma_2 - \ldots + (-1)^\lambda X_1^j \sigma_\lambda) = 0,$$

$$c_2(X_2^{\lambda+j} - X_2^{\lambda+j-1} \sigma_1 + X_2^{\lambda+j-2} \sigma_2 - \ldots + (-1)^\lambda X_2^j \sigma_\lambda) = 0,$$

$$\vdots$$

$$c_\lambda(X_\lambda^{\lambda+j} - X_\lambda^{\lambda+j-1} \sigma_1 + X_\lambda^{\lambda+j-2} \sigma_2 - \ldots + (-1)^\lambda X_\lambda^j \sigma_\lambda) = 0.$$

Summing up both sides above yields the assertion. \(\square\)
Lemma 3.2. As before, put
\[ P_k = P_k(X_1, \ldots, X_\lambda) = \sum_{i=1}^{\lambda} c_i X_i^k, \]
and define the $\lambda \times \lambda$ matrix
\[ H = H(X_1, \ldots, X_\lambda) = \begin{pmatrix} P_{\lambda-1} & P_{\lambda-2} & \cdots & P_0 \\ P_{\lambda} & P_{\lambda-1} & \cdots & P_1 \\ \vdots & \vdots & \ddots & \vdots \\ P_{2\lambda-2} & P_{2\lambda-3} & \cdots & P_{\lambda-1} \end{pmatrix}. \]
Then \( \det H = (-1)^{(\lambda-1)/2} c_1 c_2 \cdots c_\lambda \prod_{1 \leq i < j \leq \lambda} (X_i - X_j)^2 \).

Proof. It follows from the properties of Vandermonde matrices and from the fact that
\[ H = \begin{pmatrix} c_1 & c_2 & \cdots & c_\lambda \\ c_1 X_1 & c_2 X_2 & \cdots & c_\lambda X_\lambda \\ \vdots & \vdots & \ddots & \vdots \\ c_1 X_1^{\lambda-1} & c_2 X_2^{\lambda-1} & \cdots & c_\lambda X_\lambda^{\lambda-1} \end{pmatrix} \begin{pmatrix} X_1^{\lambda-1} & X_1^{\lambda-2} & \cdots & 1 \\ X_2^{\lambda-1} & X_2^{\lambda-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ X_\lambda^{\lambda-1} & X_\lambda^{\lambda-2} & \cdots & 1 \end{pmatrix}. \]

\[ \square \]

Theorem 3.3. Take any $T \subseteq AG(2, q)$ and an integer $0 < \lambda \leq (q-1)/2$. Let $E_\lambda$ denote a set of $(q-\lambda)$-uniform directions of size at most $q$. The renitent lines with slope in $E_\lambda$ are contained in an algebraic envelope of class $\lambda^2$. Furthermore, if a direction of $E_\lambda$ is not sharply $(q-\lambda)$-uniform, then the line pencil centered at that direction is fully contained in the envelope.

Proof. For each $(1 : d : 0) \in E_\lambda$ denote by $\lambda_d$ the number of renitent lines with slope $d$ and denote by $(0 : \alpha_1(d) : 1), (0 : \alpha_2(d) : 1), \ldots, (0 : \alpha_{\lambda_d}(d) : 1)$ the points of the $Y$-axis on these lines. Also, denote the typical intersection number at $(d)$ by $m_d$. Put $s := |T|$ and $T = \{(a_i : b_i : 1)\}_{i=1}^s$. Next define the polynomials
\[ \pi_k(V) := \sum_{i=1}^s (b_i - a_i V)^k \in \mathbb{F}_q[V] \]
of degree at most $k$. For any $(1 : d : 0) \in E_\lambda$ the multiset
\[ M_d := \{(b_i - a_i d)\}_{i=1}^s \]
contains $m_d$ modulo $p$ copies of $\mathbb{F}_q$ and $c_i(d) \neq 0$ modulo $p$ further copies of $\alpha_i(d)$ for $1 \leq i \leq \lambda_d$. Since $\sum_{g \in \mathbb{F}_q} g^k = 0$ for $0 \leq k \leq q-2$ and since $\pi_k(d)$ is the $k$-th power sum of $M_d$, for $0 \leq k \leq q-2$ it holds that
\[ \pi_k(d) = \sum_{i=1}^{\lambda_d} c_i(d) \alpha_i(d)^k. \] (5)
Note that $\pi_k(d)$ is as $P_k(\lambda_1, \lambda_2, \ldots, \lambda_{\lambda_d})$ in Lemma 3.1. For any integer $i \geq 0$ and $(1 : d : 0) \in E_\lambda$ put
\[ \sigma_i(d) = \sigma_i(\alpha_1(d), \ldots, \alpha_{\lambda_d}(d)). \]
For $j \geq 0, q-2 \geq \lambda_d + j$ and $(1 : d : 0) \in E_\lambda$ Lemma 3.1 yields
\[ \pi_{\lambda_d+j}(d) = \pi_{\lambda_d+j-1}(d) \sigma_1(d) - \pi_{\lambda_d+j-2}(d) \sigma_2(d) + \ldots - (-1)^j \pi_j(d) \sigma_{\lambda_d}(d). \] (6)
Define

\[ H(V) = \begin{pmatrix}
\pi_{\lambda-1}(V) & \pi_{\lambda-2}(V) & \ldots & \pi_0(V) \\
\pi_0(V) & \pi_{\lambda-1}(V) & \ldots & \pi_1(V) \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{2\lambda-2}(V) & \pi_{2\lambda-3}(V) & \ldots & \pi_{\lambda-1}(V)
\end{pmatrix}. \]

Then for each \((1 : d : 0) \in \mathcal{E}_\lambda\), since \(2\lambda - 1 \leq q - 2\), by (6) applied for \(\lambda - \lambda_d \leq j \leq 2\lambda - \lambda_d - 1\), we obtain

\[ H(d) \begin{pmatrix}
\sigma_1(d) \\
-\sigma_2(d) \\
\vdots \\
(\lambda^{\lambda+1}\sigma_\lambda(d)
\end{pmatrix} = \begin{pmatrix}
\pi_\lambda(d) \\
\pi_{\lambda+1}(d) \\
\vdots \\
\pi_{2\lambda-1}(d)
\end{pmatrix}. \quad (7)
\]

Denote by \(H_i(V)\) the matrix obtained from \(H(V)\) by replacing its \(i\)-th column with

\[ C(V) := (\pi_\lambda(V), \pi_{\lambda+1}(V), \ldots, \pi_{2\lambda-1}(V))^T. \]

By Lemma 3.2, if \((1 : d : 0) \in \mathcal{E}_\lambda\) is sharply \((q - \lambda)\)-uniform, then the determinant of \(H(d)\) is

\[ (-1)^{\lambda(\lambda-1)/2} \prod_{i=1}^\lambda c_i(d) \prod_{1 \leq i < j \leq \lambda} (\alpha_i(d) - \alpha_j(d))^2 \]

and hence \(H(d)\) is invertible.

Then by Cramer’s rule \((-1)^{i+1}\sigma_i(d) = \det H_i(d) / \det H(d)\) for \(1 \leq i \leq \lambda\).

Put \(M(V) = \det H(V)\) and \(M_i(V) = \det H_i(V)\). Then \(\deg M \leq \lambda(\lambda - 1)\) and \(\deg M_i(V) \leq \lambda^2 - \lambda + 1\). Also, \(M(d) = \det H(d)\) and \(M_i(d) = \det H_i(d)\) for \((1 : d : 0) \in \mathcal{E}_\lambda\). Consider the affine curve \(\mathcal{C}\) of degree at most \(\lambda^2\) defined by the equation

\[ f(U, V) := M(V)U^\lambda - M_1(V)U^{\lambda-1} + M_2(V)U^{\lambda-2} - \ldots - (-1)^{\lambda-1} M_{\lambda-1}(V)U + (-1)^\lambda M_\lambda(V). \]

When \(\lambda_d = \lambda\), then \((\alpha_i(d), d)\) is a point of \(\mathcal{C}\) for each \(i\) and \((1 : d : 0) \in \mathcal{E}_\lambda\). Indeed, in this case \(f(U, d)\) equals

\[ M(d)(U^\lambda - \sigma_1(d)U^{\lambda-1} + \sigma_2(d)U^{\lambda-2} - \ldots + (-1)^{\lambda-1} \sigma_{\lambda-1}(d)U + (-1)^\lambda \sigma_\lambda(d)) = M(d) \prod_{i=1}^\lambda (U - \alpha_i(d)). \]

Now consider \((1 : d : 0) \in \mathcal{E}_\lambda\) such that \(\lambda_d < \lambda\). To show that the pencil with carrier \((d)\) is contained in the dual curve it is enough to prove that \(M(d) = M_1(d) = \ldots = M_\lambda(d) = 0\). By (6) applied for \(0 \leq j \leq \lambda - 1\), in \(H(d)\) the column

\[ \begin{pmatrix}
\pi_{\lambda_d}(d) \\
\pi_{\lambda_d+1}(d) \\
\vdots \\
\pi_{\lambda_d+\lambda-1}(d)
\end{pmatrix}
\]

is the linear combination of the subsequent \(\lambda_d\) columns and hence \(\det H(d) = 0\) and so \(M(d) = 0\). By (7), the vector \(C(d)\) is the linear combination of the columns of \(H(d)\) and hence when \(H(d)\) is singular, then so does \(H_i(d)\) for each \(i\). It follows that \(\det H_i(d) = M_i(d) = 0\) as we claimed.

From the last paragraphs of the previous proof the following is clear.
Corollary 3.4. Suppose that the assumptions of the previous theorem holds and also that there exists a sharply \((q - \lambda)\)-uniform direction. Then there are at most \(\lambda^2 - \lambda\) directions incident with less than \(\lambda\) renitent lines. More precisely, if \(\lambda_d\) denotes the number of renitent lines with slope \(d\) for every \((d) \in \mathcal{E}_\lambda\), then
\[
\sum_{(d) \in \mathcal{E}_\lambda} (\lambda - \lambda_d) \leq \lambda^2 - \lambda.
\]

Proof. Suppose for the contrary that there are at least \(\lambda^2 - \lambda + 1\) directions incident with less than \(\lambda\) renitent lines. Then the algebraic envelope of renitent lines contains the pencils of carriers with these directions. It follows that the renitent lines incident with sharply \((q - \lambda)\)-uniform directions are contained in a dual curve of degree less than \(\lambda\). We know that this dual curve does not contain the corresponding pencils (in these cases \(M(d) \neq 0\) in the previous proof) and hence it cannot meet a pencil in \(\lambda\) lines, a contradiction.

To see the more precise bound, assume that \((d) \in \mathcal{E}_\lambda\) is sharply \((q - \lambda_d)\)-uniform for some \(\lambda_d < \lambda\). As at the end of the previous proof, it can be seen that each of the first \(\lambda - \lambda_d\) columns of \(H(d)\) are linear combinations of the subsequent \(\lambda_d\) columns and hence the rank of \(H(d)\) is at most \(\lambda_d\). Then for \(s > \lambda_d\) the \(s \times s\) minors of \(H(d)\) have zero determinant. It follows that \(d\) is at least a \((\lambda - \lambda_d)\)-fold root of \(\det H(V)\), see [11, Lemma 2.3], which has degree at most \(\lambda^2 - \lambda\) and the proof is completed. \(\blacksquare\)

4 The resultant method

The next result was developed in a series of papers by Szőnyi and Weiner [8, 11], see also [7] and the Appendix of [6] for the version that we cite here.

Result 4.1 (Szőnyi–Weiner Lemma). Let \(f, g \in \mathbb{F}[X, Y]\) be polynomials over the arbitrary field \(\mathbb{F}\). Assume that the coefficient of \(X^{\deg f}\) in \(f\) is not 0 and for \(y \in \mathbb{F}\) put \(k_y = \deg \gcd(f(X, y), g(X, y))\). Then for any \(y_0 \in \mathbb{F}\)
\[
\sum_{y \in \mathbb{F}} (k_y - k_{y_0})^+ \leq (\deg f - k_{y_0})\deg g - k_{y_0}). \tag{8}
\]

The main ingredient of the next proofs is how we define the polynomial \(g(X, Y)\) in the above lemma. In order to be able to detect the renitent lines at the \((q - \lambda)\)-uniform directions, in the definition of \(g\) we introduce an auxiliary polynomial \(h\) that we obtain by interpolation.

Theorem 4.2. Take a multiset \(\mathcal{M}\) of \(\mathbb{A}G(2, q)\), and fix an integer \(\lambda > 0\). Let \(\mathcal{E}_\lambda\) denote a set of \((q - \lambda)\)-uniform directions, containing at least one sharply \((q - \lambda)\)-uniform direction. If \(|\mathcal{E}_\lambda| \leq q\) then there are at least \(\lambda(|\mathcal{E}_\lambda| + 1 - \lambda)\) renitent lines and hence, at most \(\lambda^2 - \lambda\) directions incident with less than \(\lambda\) renitent lines (counted with multiplicities, as in Corollary 3.4).

Proof. Put \(|\mathcal{E}_\lambda| = k\), \(\mathcal{E}_\lambda = \{(1 : d_i : 0)\}_{i=1}^{k}\) and \(\mathcal{M} = \{(a_i : b_i : 1)\}_{i=1}^{k}\). Denote by \(m_i\) the typical intersection number corresponding to the point \((1 : d_i : 0)\). We will need the polynomial \(h(Y) := \sum_{i=1}^{k} m_i (1 - (Y - d_i)^{q-1}) \in \mathbb{F}_q[Y]\). Then \(h(d_i) = m_i\) for each \((d) \in \mathcal{E}_\lambda\). Next define the polynomials
\[
f(X, Y) := X^q - X,
g(X, Y) := \sum_i (X + a_i Y - b_i)^{q-1} - |\mathcal{M}| + h(Y).
\]

\(^1\)Here \(\alpha^+ = \max\{0, \alpha\}\). Note that \(g\) can be the zero polynomial as well, in that case \(\deg f = k_y = k_{y_0}\) and the lemma claims the trivial \(0 \leq 0\).
For a direction \((y) \in E_\lambda\) let \(\text{ind}(y)\) denote the number of renitent lines incident with \((y)\). Then for any \((y) \in E_\lambda\) it holds that

\[
k_y := \deg \gcd(f(X, y), g(X, y)) = q - \text{ind}(y).
\]

Then for any fixed \(R \in E_\lambda\), (8) gives

\[
\sum_{(y) \in E_\lambda} (\text{ind} R - \text{ind}(y)) \leq \sum_{y \in \mathbb{F}_q} (\text{ind} R - \text{ind}(y))^+ \leq \text{ind}(R)(\text{ind} R - 1).
\]

If \(\text{ind} R = \lambda\), then, as in Corollary 3.4

\[
k\lambda - \sum_{(y) \in E_\lambda} \text{ind}(y) \leq \lambda(\lambda - 1),
\]

\[
\lambda(k + 1 - \lambda) \leq \sum_{(y) \in E_\lambda} \text{ind}(y).
\]

\[\square\]

Theorem 4.3. Take a multiset \(\mathcal{M}\) of \(AG(2, q)\), \(q > 2\), and fix an integer \(\lambda > 0\). Let \(\mathcal{F}_\lambda\) denote the set of \((q - \lambda)\)-uniform directions. If \(|\mathcal{F}_\lambda| > \lambda^2 + \lambda\) then for each point \(R\) of the plane it holds that \(R\) is incident with at most \(\lambda\) or with at least \(|\mathcal{F}_\lambda| + 1 - \lambda\) renitent lines.

Proof. First we prove the following. If \(E_\lambda\) is a set of at most \(q\) directions which are \((q - \lambda)\)-uniform and \(|E_\lambda| > \lambda^2 + \lambda\), then for each point \(R\) of the plane it holds that \(R\) is incident with at most \(\lambda\) or with at least \(|E_\lambda| + 1 - \lambda\) renitent lines with slope in \(E_\lambda\).

Consider \(AG(2, q)\) embedded in \(PG(2, q)\) as \(\{(a : b : 1) : a, b \in GF(q)\}\) and apply a collineation of \(PG(2, q)\) which maps the line \([0 : 0 : 1]\) (the line at infinity) to the line \([1 : 0 : 0]\) (the \(Y\)-axis) such that \((0 : 1 : 0)\) is not in the image of \(E_\lambda\). It may be further assumed that this collineation maps \(R\) to some point \((1 : y_0 : 0)\). Denote the image of \(\mathcal{M}\) by \(\{(a_i : b_i : 1)\}_{i=1}^{|E_\lambda|} \cup \{(1 : z_j : 0)\}_{j=1}^{|E_\lambda|}\) and the image of \(E_\lambda\) by \(\{(0 : c_k : 1)\}_{k=1}^{|E_\lambda|}\).

Let \(m_k\) be the typical intersection number corresponding to the point \((0 : c_k : 1)\). We will need the polynomial \(h(X) = \sum_{k=1}^{|E_\lambda|} m_k(1 - (X - c_k)^{q-1}) \in \mathbb{F}_q[X]\). Then \(h(c_k) = m_k\) for each \(k \in \{1, \ldots, |E_\lambda|\}\). Next define the polynomials

\[
f(X, Y) := \prod_{k=1}^{|E_\lambda|} (X - c_k),
\]

\[
g(X, Y) := \sum_i (X + a_i Y - b_i)^{q-1} + \sum_j (Y - z_j)^{q-1} - |\mathcal{M}| + h(X).
\]

For a point \(P \in PG(2, q)\) let \(\text{ind} P\) denote the number of renitent lines incident with \(P\). Then for any \(y \in \mathbb{F}_q\) it holds that

\[
k_y := \deg \gcd(f(X, y), g(X, y)) = |E_\lambda| - \text{ind}(1 : y : 0).
\]

Then (8) gives

\[
\sum_{y \in \mathbb{F}_q} (\text{ind} R - \text{ind}(1 : y : 0))^+ \leq \text{ind}(q - 1 - |E_\lambda| + \text{ind} R).
\]
Note that
\[ \sum_{y \in \mathbb{F}_q} \text{ind}(1 : y : 0) \leq |\mathcal{E}_\lambda| \lambda \]
and hence
\[ q \text{ind} R - |\mathcal{E}_\lambda| \lambda \leq \text{ind}(q - 1 - |\mathcal{E}_\lambda| + \text{ind} R), \]
\[ 0 \leq \text{ind}^2 R - \text{ind}(|\mathcal{E}_\lambda| + 1) + |\mathcal{E}_\lambda| \lambda. \]
For \( \text{ind} R = \lambda + 1 \), or \( \text{ind} R = |\mathcal{E}_\lambda| - \lambda \) we have
\[ \text{ind}^2 R - \text{ind}(|\mathcal{E}_\lambda| + 1) + |\mathcal{E}_\lambda| \lambda = \lambda^2 + \lambda - |\mathcal{E}_\lambda|, \]
which is less than 0 since \( |\mathcal{E}_\lambda| > \lambda^2 + \lambda \). This proves \( \text{ind} R \leq \lambda \) or \( \text{ind} R \geq |\mathcal{E}_\lambda| - \lambda + 1 \).

Of course, if \( |\mathcal{F}_\lambda| \leq q \) then one can take \( \mathcal{E}_\lambda = \mathcal{F}_\lambda \) and this proves the theorem.

Now assume \( q + 1 = |\mathcal{F}_\lambda| > \lambda^2 + \lambda \) and take an affine point \( R \). We have to show that \( R \) is incident with at most \( \lambda \) or with at least \( q + 2 - \lambda \) renitent lines. Define \( \mathcal{E}_\lambda \) as any subset of directions of size \( q \). Note that \( q + 1 = |\mathcal{F}_\lambda| \neq \lambda^2 + \lambda + 1 \) because this would imply \( q = \lambda(\lambda + 1) \) and hence \( \lambda = 1 \) and \( q = 2 \), which we excluded. It follows that \( |\mathcal{E}_\lambda| = q > \lambda^2 + \lambda \) and hence the arguments above show that \( R \) is incident with at least \( \lambda + 1 \) renitent lines or with at least \( q + 1 - \lambda \) renitent lines. We have to exclude the cases when \( R \) is incident with exactly \( \lambda + 1 \) renitent lines or with exactly \( q + 1 - \lambda \) renitent lines.

If the former case holds, then take a direction \( S \) such that \( RS \) is not renitent and define \( \mathcal{E}_\lambda \) as \( \ell_\infty \setminus \{S\} \). Then the renitent lines incident with \( R \) have slopes in \( \mathcal{E}_\lambda \), a contradiction since there are more than \( \lambda \) but less than \( q + 1 - \lambda \) of them.

If the latter case holds, then take a direction \( S \) such that \( RS \) is renitent and, as before, define \( \mathcal{E}_\lambda \) as \( \ell_\infty \setminus \{S\} \). Then there are exactly \( q - \lambda \) renitent lines incident with \( R \) with slopes in \( \mathcal{E}_\lambda \), a contradiction since this number should be at least \( q + 1 - \lambda \) or at most \( \lambda \) (recall that \( q + 1 = |\mathcal{F}_\lambda| > \lambda^2 + \lambda \)).

With some further efforts, the earlier methods by Szönyi and Weiner [8], [11], see also [7], [4], also provide an alternative, but more laborious, route to reach similar results as in Section 3.

References


Bence Csajbók
ELKH–ELTE Geometric and Algebraic Combinatorics Research Group
ELTE Eötvös Loránd University, Budapest, Hungary
Department of Computer Science
1117 Budapest, Pázmány P. stny. 1/C, Hungary
csajbokb@cs.elte.hu

Peter Sziklai
Eötvös Loránd University, Budapest, Hungary
Department of Computer Science
1117 Budapest, Pázmány P. stny. 1/C, Hungary, and
Rényi Institute of Mathematics
1053 Budapest, Reáltanoda u. 13-15.
sziklai@cs.elte.hu

Zsuzsa Weiner
ELKH–ELTE Geometric and Algebraic Combinatorics Research Group,
1117 Budapest, Pázmány P. stny. 1/C, Hungary
zsuzsa.weiner@gmail.com
and
Prezi.com
H-1065 Budapest, Nagymező utca 54-56, Hungary