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Classes and equivalence of linear sets in $\text{PG}(1, q^n)$

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Abstract

The equivalence problem of $\mathbb{F}_q$-linear sets of rank $n$ of $\text{PG}(1, q^n)$ is investigated, also in terms of the associated variety, projecting configurations, $\mathbb{F}_q$-linear blocking sets of Rédei type and MRD-codes. We call an $\mathbb{F}_q$-linear set $L_U$ of rank $n$ in $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$ simple if for any $n$-dimensional $\mathbb{F}_q$-subspace $V$ of $W$, $L_V$ is $\text{PGL}(2, q^n)$-equivalent to $L_U$ only when $U$ and $V$ lie on the same orbit of $\Gamma \text{L}(2, q^n)$. We prove that $U = \{ (x, \text{Tr}_{q^n/q}(x)) : x \in \mathbb{F}_{q^n} \}$ defines a simple $\mathbb{F}_q$-linear set for each $n$. We provide examples of non-simple linear sets not of pseudoregulus type for $n > 4$ and we prove that all $\mathbb{F}_q$-linear sets of rank $4$ are simple in $\text{PG}(1, q^4)$.

1 Introduction

Linear sets are natural generalizations of subgeometries. Let $\Lambda = \text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(r-1, q^n)$, where $W$ is a vector space of dimension $r$ over $\mathbb{F}_{q^n}$. A point set $L$ of $\Lambda$ is said to be an $\mathbb{F}_q$-linear set of $\Lambda$ of rank $k$ if it is defined by the non-zero vectors of a $k$-dimensional $\mathbb{F}_q$-vector subspace $U$ of $W$, i.e.,

$$L = L_U = \{ (u)_{\mathbb{F}_{q^n}} : u \in U \setminus \{0\} \}.$$

The maximum field of linearity of an $\mathbb{F}_q$-linear set $L_U$ is $\mathbb{F}_t$ if $t \mid n$ is the largest integer such that $L_U$ is an $\mathbb{F}_t$-linear set. In the recent years, starting from the paper [19] by Lunardon, linear sets have been used to construct or characterize various objects in finite geometry, such as blocking sets and multiple blocking sets in finite projective spaces, two-intersection sets in finite projective spaces, translation spreads of the Cayley Generalized Hexagon.

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translation ovoids of polar spaces, semifield flocks and finite semifields. For a survey on linear sets we refer the reader to [26], see also [15].

One of the most natural questions about linear sets is their equivalence. Two linear sets \( L_U \) and \( L_V \) of \( \text{PG}(r-1,q^n) \) are said to be \( \text{PTL-equivalent} \) (or simply \( \text{equivalent} \)) if there is an element \( \varphi \) in \( \text{PTL}(r,q^n) \) such that \( L_U^\varphi = L_V \). In the applications it is crucial to have methods to decide whether two linear sets are equivalent or not. For \( f \in \text{GL}(r,q^n) \) we have \( L_U^f = L_{U^f} \), where \( \varphi_f \) denotes the collineation of \( \text{PG}(W,F_{q^n}) \) induced by \( f \). It follows that if \( U \) and \( V \) are \( F_q \)-subspaces of \( W \) belonging to the same orbit of \( \text{GL}(r,q^n) \), then \( L_U \) and \( L_V \) are equivalent. The above condition is only sufficient but not necessary to obtain equivalent linear sets. This follows also from the fact that \( F_q \)-subspaces of \( W \) with different ranks can define the same linear set, for example \( F_q \)-linear sets of \( \text{PG}(r-1,q^n) \) of rank \( k \geq rn-n+1 \) are all the same: they coincide with \( \text{PG}(r-1,q^n) \). As it was showed recently in [7], if \( r = 2 \), then there exist \( F_q \)-subspaces of \( W \) of the same rank but on different orbits of \( \text{GL}(2,q^n) \) defining the same linear set of \( \text{PG}(1,q^n) \).

This observation motivates the following definition. An \( F_q \)-linear set \( L_U \) with maximum field of linearity \( F_q \) is called \( \text{simple} \) if for each \( F_q \)-subspace \( V \) of \( W \) with \( \dim_q(U) = \dim_q(V) \), \( L_U = L_V \) only if \( U \) and \( V \) are in the same orbit of \( \text{GL}(2,q^n) \) or, equivalently, if for each \( F_q \)-subspace \( V \) of \( W \) with \( \dim_q(U) = \dim_q(V) \), \( L_U \) is \( \text{PTL}(2,q^n) \)-equivalent to \( L_V \) only if \( U \) and \( V \) are in the same orbit of \( \text{GL}(2,q^n) \).

Natural examples of simple linear sets are the subgeometries (cf. [18, Theorem 2.6] and [14, Section 25.5]). In [6] it was proved that \( F_q \)-linear sets of rank \( n+1 \) of \( \text{PG}(2,q^n) \) admitting \( (q+1) \)-secants are simple. This allowed the authors to translate the question of equivalence to the study of the orbits of the stabilizer of a subgeometry on subspaces and hence to obtain the complete classification of \( F_q \)-linear blocking sets in \( \text{PG}(2,q^4) \). Until now, the only known examples of non-simple linear sets are those of pseudoregulus type of \( \text{PG}(1,q^n) \) for \( n \geq 5 \) and \( n \neq 6 \), see [7].

In this paper we focus on linear sets of rank \( n \) of \( \text{PG}(1,q^n) \). We first introduce a method which can be used to find non-simple linear sets of rank \( n \) of \( \text{PG}(1,q^n) \). Let \( L_U \) be a linear set of rank \( n \) of \( \text{PG}(W,F_{q^n}) = \text{PG}(1,q^n) \) and let \( \beta \) be a non-degenerate alternating form of \( W \). Denote by \( \perp \) the orthogonal complement map induced by \( \text{Tr}_{q^n/q} \circ \beta \) on \( W \) (considered as an \( F_q \)-vector space). Then \( U \) and \( U^\perp \) defines the same linear set (cf. Result 2.1) and if \( U \) and \( U^\perp \) lie on different orbits of \( \text{GL}(W,F_{q^n}) \), then \( L_U \) is non-simple. Using this approach we show that there are non-simple linear sets of rank \( n \) of \( \text{PG}(1,q^n) \) for \( n \geq 5 \), not of pseudoregulus type (cf. Proposition
3.10). Contrary to what we expected initially, simple linear sets are harder to find than non-simple linear sets. We prove that the linear set of PG(1, q^n) defined by the trace function is simple (cf. Theorem 3.7). We also show that linear sets of rank n of PG(1, q^n) are simple for n ≤ 4 (cf. Theorem 4.5).

Moreover, in PG(1, q^n) we extend the definition of simple linear sets and introduce the Z(FL)-class and the GL-class for linear sets of rank n. In Section 5 we point out the meaning of these classes in terms of equivalence of the associated blocking sets, MRD-codes and projecting configurations.

2 Definitions and preliminary results

2.1 Dual linear sets with respect to a symplectic polarity of a line

For α ∈ \mathbb{F}_{q^n} and a divisor h of n we will denote by Tr_{q^n/q^h}(α) the trace of α over the subfield \mathbb{F}_{q^h}, that is, Tr_{q^n/q^h}(α) = α + αq^h + \ldots + αq^{n-h}. By N_{q^n/q^h}(α) we will denote the norm of α over the subfield \mathbb{F}_{q^h}, that is, N_{q^n/q^h}(α) = α^{1+q^h+\ldots+q^{n-h}}. Since in the paper we will use only norms over \mathbb{F}_q, the function N_{q^n/q} will be denoted simply by N.

Starting from a linear set \mathcal{L}_U of PG(r, q^n) and using a polarity \tau of the space it is always possible to construct another linear set, which is called dual linear set of \mathcal{L}_U with respect to the polarity \tau (see [26]). In particular, let \mathcal{L}_U be an \mathbb{F}_q-linear set of rank n of a line PG(W, \mathbb{F}_{q^n}) and let \beta : W \times W \rightarrow \mathbb{F}_{q^n} be a non-degenerate reflexive \mathbb{F}_{q^n}-sesquilinear form on the 2-dimensional vector space W over \mathbb{F}_{q^n} determining a polarity \tau. The map Tr_{q^n/q} \circ \beta is a non-degenerate reflexive \mathbb{F}_{q^n}-sesquilinear form on W, when W is regarded as a 2n-dimensional vector space over \mathbb{F}_q. Indeed, suppose the contrary, then there exists an element v ∈ W such that Tr_{q^n/q}(\beta(u, v)) = 0 for each u ∈ W. Then the image of \varphi: u ∈ W \mapsto \beta(u, v) ∈ \mathbb{F}_{q^n} is contained in the kernel of Tr_{q^n/q}, and hence the rank of \varphi over \mathbb{F}_q is at most n – 1. On the other hand, since \beta is non-degenerate and \varphi is \mathbb{F}_{q^n}-linear, we have rk_{\mathbb{F}_{q^n}} \varphi = 1 and rk_{\mathbb{F}_{q^n}} \varphi = n, a contradiction.

Let \perp_\beta and \perp_\beta^\perp be the orthogonal complement maps defined by \beta and Tr_{q^n/q} \circ \beta on the lattices of the \mathbb{F}_{q^n}-subspaces and \mathbb{F}_{q^n}-subspaces of W, respectively. The dual linear set of \mathcal{L}_U with respect to the polarity \tau is the \mathbb{F}_{q^n}-linear set of rank n of PG(W, \mathbb{F}_{q^n}) defined by the orthogonal complement \mathcal{U}_U^\perp and it will be denoted by \mathcal{L}_U. Also, up to projective equivalence, such a linear set does not depend on \tau [26, Proposition 2.5].
For a point \( P = (z)_{\mathbb{F}_{q^n}} \in \text{PG}(W, \mathbb{F}_{q^n}) \) the weight of \( P \) with respect to the linear set \( L_U \) is \( w_{L_U}(P) := \dim_q ((z)_{\mathbb{F}_{q^n}} \cap U) \).

**Result 2.1.** From [26, Property 2.6] (with \( r = 2, s = 0 \) and \( t = n \)) it can be easily seen that if \( L_U \) is an \( \mathbb{F}_q \)-linear set of rank \( n \) of a line \( \text{PG}(1, q^n) \) and \( L_U^{t} \) is its dual linear set with respect to a polarity \( \tau \), then \( w_{L_U^{t}}(P^{t}) = w_{L_U}(P) \) for each point \( P \in \text{PG}(1, q^n) \). If \( \tau \) is a symplectic polarity of a line \( \text{PG}(1, q^n) \), then \( P^{\tau} = P \) and hence \( L_U = L_U^{t} = L_U^{t, \tau} \).

### 2.2 \( \mathbb{F}_q \)-linear sets of \( \text{PG}(1, q^n) \) of class \( r \)

In this paper we investigate the equivalence of \( \mathbb{F}_q \)-linear sets of rank \( n \) of the projective line \( \text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n) \). The first step is to determine the \( \mathbb{F}_q \)-vector subspaces of \( W \) defining the same linear set. This motivates the definition of the \( Z(\Gamma L) \)-class and \( \Gamma L \)-class of a linear set \( L_U \) of \( \text{PG}(1, q^n) \) (cf. Definitions 2.4 and 2.5). The next proposition relies on the characterization of functions over \( \mathbb{F}_q \) determining few directions. It states that the \( \mathbb{F}_q \)-rank of \( L_U \) of \( \text{PG}(1, q^n) \) is uniquely defined when the maximum field of linearity of \( L_U \) is \( \mathbb{F}_q \). This will allow us to state our definitions and results without further conditions on the rank of the corresponding \( \mathbb{F}_q \)-subspaces.

For an \( \mathbb{F}_q \) to \( \mathbb{F}_q \) function \( f \), the set of directions determined by \( f \) is

\[
D_f := \left\{ \frac{f(x) - f(y)}{x - y} : x, y \in \mathbb{F}_{q^n}, x \neq y \right\}.
\]

**Theorem 2.2** (Ball et al. [3] and Ball [1]). Let \( f \) be a function from \( \mathbb{F}_q \) to \( \mathbb{F}_q \), \( q = p^h \), and let \( N \) be the number of directions determined by \( f \). Let \( s = p^e \) be maximal such that any line with a direction determined by \( f \) that is incident with a point of the graph of \( f \) is incident with a multiple of \( s \) points of the graph of \( f \). Then one of the following holds.

1. \( s = 1 \) and \( (q + 3)/2 \leq N \leq q + 1 \),
2. \( e|h \), \( q/s + 1 \leq N \leq (q - 1)/(s - 1) \),
3. \( s = q \) and \( N = 1 \).

Moreover if \( s > 2 \), then the graph of \( f \) is \( \mathbb{F}_q \)-linear.

**Proposition 2.3.** Let \( L_U \) be an \( \mathbb{F}_q \)-linear set of \( \text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n) \) of rank \( n \). The maximum field of linearity of \( L_U \) is \( \mathbb{F}_{q^d} \), where

\[
d = \min\{w_{L_U}(P) : P \in L_U\}.
\]
If the maximum field of linearity of $L_U$ is $\mathbb{F}_q$, then the rank of $L_U$ as an $\mathbb{F}_q$-linear set is uniquely defined, i.e. for each $\mathbb{F}_q$-subspace $V$ of $W$ if $L_U = L_V$, then $\dim_q(V) = n$.

Proof. Since the action of $\Gamma L(2,q^n)$ preserves the maximum field of linearity and the weight of points, we can assume, up to the action of $\Gamma L(2,q^n)$, that $U = \{(x,f(x)) : x \in \mathbb{F}_{q^n}\}$ for some $q$-polynomial $f$ over $\mathbb{F}_{q^n}$. Since $f$ is linear, $|L_U|$ is the size of the set of directions determined by $f$. Also, a line $\ell$ with slope $m$ meets the graph of $f$ in $q^d$ points, where $t = w_{L_U}((1,m)\mathbb{F}_{q^n})$, i.e. $\left\{|z \in \mathbb{F}_{q^n}^* : f(z)/z = m\right\} = q^d - 1$.

Let $d = \min\{w_{L_U}(P) : P \in L_U\}$. If $q = p^r$, $p$ prime, then $p^{dr}$ is the largest $p$-power such that every line with a determined direction that meets the graph of $f$ meets the graph of $f$ in a multiple of $s = p^{dr}$ points. Then Theorem 2.2 yields that either $s = q^n$ and $f(x) = \lambda x$ for some $\lambda \in \mathbb{F}_{q^n}$, or $\mathbb{F}_{q^n}$ is a proper subfield of $\mathbb{F}_q^n$ and

$$q^{n-d} + 1 \leq |D_f| \leq \frac{q^n - 1}{q^d - 1}. \tag{1}$$

Moreover, if $q^d > 2$, then $f$ is $\mathbb{F}_{q^n}$-linear. In our case we already know that $f$ is $\mathbb{F}_q$-linear, so even in the case $q^d = 2$ it follows that $U$ is an $\mathbb{F}_q$-subspace of $W$ and hence $L_U$ is an $\mathbb{F}_q$-linear set.

We show that $\mathbb{F}_{q^n}$ is the maximum field of linearity of $L_U$. Suppose, contrary to our claim, that $L_U$ is $\mathbb{F}_q$-linear of rank $z$ for some $r > d$. Then $L_U$ is also $\mathbb{F}_q$-linear of rank $rz$. It follows that $rz \leq n$ since otherwise $L_U = \text{PG}(1,q^n)$. Then for the size of $L_U$ we get $|L_U| \leq \frac{(q^{rz} - 1)/(q^r - 1)}{(q^{n} - 1)/(q - 1)}$, and this number turns out to be less than the lower bound in (1). This shows $r = d$.

Now suppose that $\mathbb{F}_q$ is the maximum field of linearity of $L_U$ and let $V$ be an $r$-dimensional $\mathbb{F}_q$-subspace of $W$ such that $L_U = L_V$. We cannot have $r > n$ since $L_U \neq \text{PG}(1,q^n)$. Suppose, contrary to our claim, that $r \leq n - 1$. Then $|L_U| \leq \frac{(q^{n-1} - 1)/(q - 1)}{|L_U|}$ contradicting (1) which gives $q^{n-1} + 1 \leq |L_U|$. This concludes the proof. \hfill $\Box$

Now we can give the following definitions of classes of an $\mathbb{F}_q$-linear set of a line.

**Definition 2.4.** Let $L_U$ be an $\mathbb{F}_q$-linear set of $\text{PG}(W,\mathbb{F}_{q^n}) = \text{PG}(1,q^n)$ of rank $n$ with maximum field of linearity $\mathbb{F}_q$. We say that $L_U$ is of $\mathcal{Z}(\Gamma L)$-class $r$ if $r$ is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1, U_2, \ldots, U_r$ of $W$ with $L_{U_i} = L_U$ for $i \in \{1,2,\ldots,r\}$ and $U_i \neq \lambda U_j$ for each $\lambda \in \mathbb{F}_{q^n}$ and for each $i \neq j$, $i,j \in \{1,2,\ldots,r\}$.\hfill $\Box$
Definition 2.5. Let $L_U$ be an $\mathbb{F}_q$-linear set of $\text{PG}(W, \mathbb{F}_q^n) = \text{PG}(1, q^n)$ of rank $n$ with maximum field of linearity $\mathbb{F}_q$. We say that $L_U$ is of $\Gamma L$-class $s$ if $s$ is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1, U_2, \ldots, U_s$ of $W$ with $L_{U_i} = L_U$ for $i \in \{1, 2, \ldots, s\}$ and there is no $f \in \Gamma L(2, q^n)$ such that $U_i = U_j^f$ for each $i \neq j, i, j \in \{1, 2, \ldots, s\}$.

Simple linear sets (cf. Section 1) of $\text{PG}(1, q^n)$ are exactly those of $\Gamma L$-class one. The next proposition is easy to show.

Proposition 2.6. Let $L_U$ be an $\mathbb{F}_q$-linear set of $\text{PG}(1, q^n)$ of rank $n$ with maximum field of linearity $\mathbb{F}_q$ and let $\varphi$ be a collineation of $\text{PG}(1, q^n)$. Then $L_U$ and $L_U^\varphi$ have the same $\mathbb{Z}(\Gamma L)$-class and $\Gamma L$-class. \qed

Remark 2.7. Let $L_U$ be an $\mathbb{F}_q$-linear set of rank $n$ of $\text{PG}(1, q^n)$ with $\Gamma L$-class $s$ and let $U_1, U_2, \ldots, U_s$ be $\mathbb{F}_q$-subspaces belonging to different orbits of $\Gamma L(2, q^n)$ and defining $L_U$. The $\mathbb{Z}(\Gamma L)$-orbit of $L_U$ is the set
\[
\bigcup_{i=1}^s \{L_{U_i^f} : f \in \Gamma L(2, q^n)\}.
\]

3 Examples of simple and non-simple linear sets of $\text{PG}(1, q^n)$

Let $L_U$ be an $\mathbb{F}_q$-linear set of rank $n$ of $\text{PG}(1, q^n)$. We can always assume (up to a projectivity) that $L_U$ does not contain the point $((0, 1))_{\mathbb{F}_q^n}$. Then $U = U_f = \{(x, f(x)) : x \in \mathbb{F}_q^n\}$, for some $q$-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ over $\mathbb{F}_q^n$. For the sake of simplicity we will write $L_f$ instead of $L_{U_f}$ to denote the linear set defined by $U_f$.

According to Result 2.1 and using the same notations as in Section 2.1 if $L_U$ is an $\mathbb{F}_q$-linear set of rank $n$ of $\text{PG}(1, q^n)$ and $\tau$ is a symplectic polarity, then $U^{\perp_\tau}$ defines the same linear set as $U$. Since in general $U^{\perp_\tau}$ and $U$ are not equivalent under the action of the group $\Gamma L(2, q^n)$, some linear sets of a line are harder to find than non-simple linear sets.

Consider the non-degenerate symmetric bilinear form of $\mathbb{F}_q^n$ over $\mathbb{F}_q$ defined by the following rule
\[
< x, y > := \text{Tr}_{q^n/q}(xy).
\] (2)

Then the adjoint map $f$ of an $\mathbb{F}_q$-linear map $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ of $\mathbb{F}_q^n$ (with
respect to the bilinear form $\langle \cdot, \cdot \rangle$ is

\[ \hat{f}(x) := \sum_{i=0}^{n-1} a_i^q x^{q^n-i}. \]  

(3)

Let $\eta: \mathbb{F}_q^2 \times \mathbb{F}_q^n \to \mathbb{F}_q^n$ be the non-degenerate alternating bilinear form of $\mathbb{F}_q^2$ defined by

\[ \eta((x, y), (u, v)) = xv - yu. \]  

(4)

Then $\eta$ induces a symplectic polarity on the line $\text{PG}(1, q^n)$ and

\[ \eta'((x, y), (u, v)) = \text{Tr}_{q^n/q}(\eta((x, y), (u, v))) \]  

(5)

is a non-degenerate alternating bilinear form on $\mathbb{F}_q^2$, when $\mathbb{F}_q^2$ is regarded as a $2n$-dimensional vector space over $\mathbb{F}_q$. We will always denote in the paper by $\perp$ and $\perp'$ the orthogonal complement maps defined by $\eta$ and $\eta'$ on the lattices of the $\mathbb{F}_{q^n}$-subspaces and the $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^2$, respectively. Direct calculation shows that

\[ U_f^\perp = U_f. \]  

(6)

Result 2.1 and (6) allow us to slightly reformulate [4, Lemma 2.6].

**Lemma 3.1** ([4]). Let $L_f = \{ \langle (x, f(x)) \rangle_{\mathbb{F}_q^n} : x \in \mathbb{F}_q^n \}$ be an $\mathbb{F}_q$-linear set of $\text{PG}(1,q^n)$ of rank $n$, with $f(x)$ a $q$-polynomial over $\mathbb{F}_q^n$, and let $\hat{f}$ be the adjoint of $f$ with respect to the bilinear form (2). Then for each point $P \in \text{PG}(1,q^n)$ we have $w_{L_f}(P) = w_{L_{\hat{f}}}(P)$. In particular, $L_f = L_{\hat{f}}$ and the maps defined by $f(x)/x$ and $\hat{f}(x)/x$ have the same image.

**Lemma 3.2.** Let $\varphi$ be an $\mathbb{F}_q$-linear map of $\mathbb{F}_q^n$ and for $\lambda \in \mathbb{F}_{q^n}$ let $\varphi_\lambda$ denote the $\mathbb{F}_q$-linear map: $x \mapsto \varphi(\lambda x)/\lambda$. Then for each point $P \in \text{PG}(1,q^n)$ we have $w_{L_\varphi}(P) = w_{L_{\varphi_\lambda}}(P)$. In particular, $L_\varphi = L_{\varphi_\lambda}$.

Proof. The statements follow from $\lambda U_{\varphi_\lambda} = U_{\varphi}$.

**Remark 3.3.** The results of Lemmas 3.1 and 3.2 can also be obtained via Dickson matrices. For a $q$-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ over $\mathbb{F}_q^n$ let $D_f$ denote the associated Dickson matrix (or $q$-circulant matrix)

\[ D_f := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1^q & a_0^q & \cdots & a_{n-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}^{q^{n-1}} & a_{n-2}^{q^{n-1}} & \cdots & a_0^{q^{n-1}} \end{pmatrix}. \]
When \( f(x) = \lambda x \) for some \( \lambda \in \mathbb{F}_{q^n} \) we will simply write \( D_\lambda \). The rank of the matrix \( D_f \) equals the rank of the \( \mathbb{F}_q \)-linear map \( f \), see for example [28]. We will denote the point \( \langle (1, \lambda) \rangle_{q^n} \) by \( P_\lambda \).

Transposition preserves the rank of matrices and \( D_f^T = D_f, \ D_\lambda^T = D_\lambda \). It follows that

\[
\dim_q \ker(D_f - D_\lambda) = \dim_q \ker(D_f - D_\lambda)^T = \dim_q \ker(D_f - D_\lambda),
\]

and hence for each \( \lambda \in \mathbb{F}_{q^n} \) we have \( w_{L_f}(P_\lambda) = w_{L_f}(P_\lambda) \).

Let \( f_\mu(x) = f(x^\mu) / \mu \). It is easy to see that \( D_{1/\mu} D_f D_\mu = D_{f_\mu} \) and

\[
\dim_q \ker(D_f - D_\lambda) = \dim_q \ker D_{1/\mu}(D_f - D_\lambda) D_\mu = \dim_q \ker(D_{f_\mu} - D_\lambda),
\]

and hence \( w_{L_f}(P_\lambda) = w_{L_{f_\mu}}(P_\lambda) \) for each \( \lambda \in \mathbb{F}_{q^n} \).

From the previous arguments it follows that linear sets \( L_f \) with \( f(x) = \hat{f}(x) \) are good candidates for being simple. In the next section we show that the trace function, which has the previous property, defines a simple linear set. We are going to use the following lemmas which will also be useful later.

**Lemma 3.4.** Let \( f \) and \( g \) be two linearized polynomials. If \( L_f = L_g \), then for each positive integer \( d \) the following holds

\[
\sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{f(x)}{x} \right)^d = \sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{g(x)}{x} \right)^d.
\]

**Proof.** If \( L_f = L_g = L \), then \( \{ f(x)/x : x \in \mathbb{F}_{q^n}^* \} = \{ g(x)/x : x \in \mathbb{F}_{q^n}^* \} =: H \). For each \( h \in H \) we have \( |\{ x : f(x)/x = h \}| = q^i - 1 \), where \( i \) is the weight of the point \( \langle (1, h) \rangle_{q^n} \in L \) w.r.t. \( U_f \), and similarly \( |\{ x : g(x)/x = h \}| = q^j - 1 \), where \( j \) is the weight of the point \( \langle (1, h) \rangle_{q^n} \in L \) w.r.t. \( U_g \). Because of the characteristic of \( \mathbb{F}_{q^n} \), we obtain:

\[
\sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{f(x)}{x} \right)^d = - \sum_{h \in H} h^d = \sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{g(x)}{x} \right)^d.
\]

\( \square \)

For the sake of completeness we give a proof of the following well-known result.

**Lemma 3.5.** For any prime power \( q \) and integer \( d \) we have \( \sum_{x \in \mathbb{F}_q^*} x^d = -1 \) if \( q - 1 | d \) and \( \sum_{x \in \mathbb{F}_q^*} x^d = 0 \) otherwise.

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Proof. Let $g$ denote a primitive element of $\mathbb{F}_q$ and put $s = \sum_{i=0}^{n-2} g^i$. Then $sg^d = s$ and hence either $s = 0$, or $g^d = 1$. In the latter case $q - 1 \mid d$ since $g$ was a primitive element and hence $x^d = 1$ for each $x \in \mathbb{F}_q$.

**Lemma 3.6.** Let $f(x) = \sum_{i=0}^{n-1} a_i x^i$ and $g(x) = \sum_{i=0}^{n-1} b_i x^i$ be two $q$-polynomials over $\mathbb{F}_q^n$, such that $L_f = L_g$. Then

$$a_0 = b_0,$$

and for $k = 1, 2, \ldots, n - 1$ it holds that

$$a_k a_{n-k} = b_k b_{n-k},$$

for $k = 2, 3, \ldots, n - 1$ it holds that

$$a_1 a_{k-1} a_{n-k} + a_k a_{n-k-1} a_{n-k} = b_1 b_{k-1} b_{n-k} + b_k b_{n-k-1} b_{n-k+1}.$$

**Proof.** We are going to use Lemma 3.5 together with Lemma 3.4 with different choices of $d$.

With $d = 1$ we have

$$\sum_{x \in \mathbb{F}_q^n} \sum_{i=0}^{n-1} a_i x^i - 1 = \sum_{i=0}^{n-1} a_i x^i - 1,$$

and hence

$$\sum_{i=0}^{n-1} a_i \sum_{x \in \mathbb{F}_q^n} x^i - 1 = \sum_{i=0}^{n-1} b_i \sum_{x \in \mathbb{F}_q^n} x^i - 1.$$

Since $q^n - 1$ cannot divide $q^i - 1$ with $i = 1, 2, \ldots, n - 1$, $a_0 = b_0 = c$ follows. Let $\varphi$ denote the $\mathbb{F}_q^n$-linear map which fixes $(0, 1)$ and maps $(1, 0)$ to $(1, -c)$. Then $U_\varphi f = U_\varphi f$ and $U_\varphi g = U_\varphi g'$ with $f' = \sum_{i=0}^{n-1} a_i x^i$, $g' = \sum_{i=0}^{n-1} b_i x^i$ and of course with $L_{f'} = L_{g'}$. It follows that we may assume $c = 0$.

First we show that (8) holds. With $d = q^k + 1$, $1 \leq k \leq n - 1$ we obtain

$$\sum_{1 \leq i, j \leq n-1} a_i a_j^{q^k} \sum_{x \in \mathbb{F}_q^n} x^{q^i+1+q^j+q^k-q^k} = \sum_{1 \leq i, j \leq n-1} b_i b_j^{q^k} \sum_{x \in \mathbb{F}_q^n} x^{q^i+1+q^j+q^k-q^k}.$$ 

$\sum_{x \in \mathbb{F}_q^n} x^{q^i+1+q^j+q^k-q^k} = -1$ if and only if $q^i+q^{j+k} \equiv q^k+1 \pmod{q^n-1}$, and zero otherwise. Suppose that the former case holds.

First consider $j + k \leq n - 1$. Then $q^i+q^{j+k} \leq q^{n-1}+q^{n-1} < q^k+1 + 2(q^n-1)$ hence one of the following holds.
• If \( q^i + q^{i+k} = q^k + 1 \), then the right hand side is not divisible by \( q \), a contradiction.

• If \( q^i + q^{i+k} = q^k + 1 + (q^n - 1) = q^n + q^k \), then \( j+k = n \), a contradiction.

Now consider the case \( j + k \geq n \). Then \( q^i + q^{i+k} \equiv q^i + q^{i+k-n} \equiv q^k + 1 \pmod{q^n - 1} \). Since \( j + k \leq 2(n - 1) \), we have \( q^i + q^{i+k-n} \leq q^{n-1} + q^{n-2} < q^k + 1 + 2(q^n - 1) \), hence one of the following holds.

• If \( q^i + q^{i+k-n} = q^k + 1 \), then \( j + k = n \) and \( i = k \).

• If \( q^i + q^{i+k-n} = q^k + 1 + (q^n - 1) = q^n + q^k \), then there is no solution since \( j + k - n \notin \{k, n\} \).

Hence (8) follows. Now we show that (9) also holds. Note that in this case \( n \geq 3 \), otherwise there is no \( k \) with \( 2 \leq k \leq n - 1 \). With \( d = q^k + q + 1 \), we obtain

\[
\sum_{1 \leq i, j, m \leq n-1} a_i a_j a_m^k \sum_{x \in \mathbb{F}_q} x^{q^i-1+q^{j+1}+q^{m+k}-q^k} = \\
\sum_{1 \leq i, j, m \leq n-1} b_i b_j b_m^k \sum_{x \in \mathbb{F}_q} x^{q^i-1+q^{j+1}+q^{m+k}-q^k}.
\]

\[
\sum_{x \in \mathbb{F}_q} x^{q^i-1+q^{j+1}+q^{m+k}-q^k} = -1 \text{ if and only if } q^i+q^{j+1}+q^{m+k} \equiv q^k+q+1 \pmod{q^n - 1}, \text{ and zero otherwise. Suppose that the former case holds.}
\]

First consider \( m+k \leq n-1 \). Then \( q^i+q^{j+1}+q^{m+k} \leq q^{n-1} + q^n + q^{n-1} < q^k + q + 1 + 2(q^n - 1) \) hence one of the following holds.

• If \( q^i+q^{j+1}+q^{m+k} = q^k + q + 1 \), then the right hand side is not divisible by \( q \), a contradiction.

• If \( q^i+q^{j+1}+q^{m+k} = q^k + q + 1 + (q^n - 1) = q^n + q^k + q \), then \( m+k = n \), \( j+1 = k \) and \( i = 1 \), a contradiction.

Now consider the case \( m+k \geq n \). Then \( q^i+q^{j+1}+q^{m+k} \equiv q^i+q^{j+1}+q^{m+k-n} \equiv q^k + q + 1 \pmod{q^n - 1} \). We have \( q^i+q^{j+1}+q^{m+k-n} \leq q^{n-1} + q^n + q^{n-2} < q^k + q + 1 + 2(q^n - 1) \) hence one of the following holds.

• If \( q^i+q^{j+1}+q^{m+k-n} = q^k + q + 1 \), then \( j+1 = k \), \( i = 1 \) and \( m+k = n \).

• If \( q^i+q^{j+1}+q^{m+k-n} = q^k + q + 1 + (q^n - 1) = q^n + q^k + q \), then \( j+1 = n \), \( i = k \) and \( m+k = n+1 \).

This concludes the proof.
3.1 Linear sets defined by the trace function

We show that there exist at least one simple $\mathbb{F}_q$-linear set in $\text{PG}(1, q^n)$ for each $q$ and $n$. Let $V = \{(x, \text{Tr}_{q^n/\mathbb{F}_q}(x)) : x \in \mathbb{F}_{q^n}\}$. We show that $L_U = L_V$ occurs for an $\mathbb{F}_q$-subspace $U$ of $W$ if and only if $V = \lambda U$ for some $\lambda \in \mathbb{F}_{q^n}^*$, i.e. $L_V$ is of $Z(\text{GL})$-class one and hence simple.

**Theorem 3.7.** Let $V = \{(x, \text{Tr}_{q^n/\mathbb{F}_q}(x)) : x \in \mathbb{F}_{q^n}\}$, then the $\mathbb{F}_q$-linear set $L_V$ of $\text{PG}(1, q^n)$ is of $Z(\text{GL})$-class one.

**Proof.** Suppose $L_{U_f} = L_V$ with $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ and $f(x) = \sum_{i=0}^{n-1} a_i x^i$. We are going to use Lemma 3.6 with $g(x) = \text{Tr}_{q^n/\mathbb{F}_q}(x)$. The coefficients $b_0, b_1, \ldots, b_{n-1}$ of $g(x)$ are 1, hence $a_0 = 1$, and for $k = 1, 2, \ldots, n-1$

$$a_k a_{n-k}^q = 1,$$

for $k = 2, 3, \ldots, n-1$

$$a_1 a_{k-1}^q a_{n-k}^q + a_k a_{n-1}^q a_{n-k+1}^q = 2. \tag{11}$$

Note that (10) implies $a_i \neq 0$ for $i = 1, 2, \ldots, n-1$. First we prove

$$a_i = a_1^{1+q+\ldots+q^{i-1}} \tag{12}$$

by induction on $i$ for each $0 < i < n$. The assertion holds for $i = 1$. Suppose that it holds for some integer $i - 1$ with $1 < i < n$. We prove that it also holds for $i$. Then (11) with $k = i$ gives

$$a_1 a_{i-1}^q a_{n-i}^q + a_i a_{n-1}^q a_{n-i+1}^q = 2. \tag{13}$$

Also, (10) with $k = i$, $k = i - 1$ and $k = 1$, respectively, gives

$$a_{n-i}^q = 1/a_i,$$

$$a_{n-i+1}^q = 1/a_{i-1}^q,$$

$$a_{n-1}^q = 1/a_1.$$  

Then (13) gives

$$a_1 a_{i-1}^q/a_i + a_i/(a_1 a_{i-1}^q) = 2. \tag{14}$$

It follows that $a_1 a_{i-1}^q/a_i = 1$ and hence the induction hypothesis on $a_{i-1}$ yields $a_i = a_1^{1+q+\ldots+q^{i-1}}$. 

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Finally we show $N(a_1) = 1$. First consider $n$ even. Then (10) with $k = n/2$ gives $a_{n/2}^{q^{n/2}+1} = 1$. Applying (12) yields $N(a_1) = 1$. If $n$ is odd, then (10) with $k = (n - 1)/2$ gives $a_{(n-1)/2}^{q^{(n-1)/2}} = 1$. Applying (12) yields $N(a_1) = 1$. It follows that $a_1 = \lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^n}$ and hence $f(x) = \sum_{i=0}^{n-1} \lambda^{q-1}x^i$. Then $\lambda U_f = \{ (x, \text{Tr}_{q^n/q}(x)) : x \in \mathbb{F}_{q^n}^* \}$. \hfill \(\square\)

**Remark 3.8.** We point out that in the above theorem we do not have any assumption on the weight of points of $L_U$. In the special case when $L_U = L_V$ and $L_U$ has a point of weight $n-1$, then the $\text{GL}(2, q^n)$-equivalence of $U$ and $V$ can be deduced also from [8, Theorem 2.3].

### 3.2 Non-simple linear sets

An $\mathbb{F}_q$-linear set of *pseudoregulus type* of $\text{PG}(1, q^n)$ is any linear set equivalent to $\{ ((x, x^q))_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^* \}$. In [7] it was proved that the $\Gamma L$-class of such linear sets is $\varphi(n)/2$, hence they are non-simple for $n = 5$ and $n > 6$. So far, these are the only known non-simple linear sets of $\text{PG}(1, q^n)$. Here we show that $\mathbb{F}_q$-linear sets $L_f$ of $\text{PG}(1, q^n)$ introduced by Lunardon and Polverino, which are not of pseudoregulus type ([22, Theorems 2 and 3]), are non-simple as well. Let us start by proving the following preliminary result.

**Proposition 3.9.** Let $f(x) = \sum_{i=0}^{n-1} a_i x^i$. There is an $\mathbb{F}_q$-semilinear map between $U_f$ and $U_f$ if and only if the following system of $n$ equations has a solution $A, B, C, D \in \mathbb{F}_q$, $AD - BC \neq 0$, $\sigma = p^k$:

$$C + Da_i^\sigma - a_0A = \sum_{i=0}^{n-1} (Ba_i a_i^\sigma)^{q^{n-i}} ,$$

$$\ldots$$

$$Da_m^\sigma - (a_{n-m}A)^{q^m} = \sum_{i=0}^{n-1} (Ba_i a_i^\sigma)^{q^{n-i}} , \text{ with } m = 1, \ldots, n-2,$$

$$\ldots$$

$$Da_{n-1}^\sigma - (a_1A)^{q^{n-1}} = \sum_{i=0}^{n-1} (Ba_i a_i^\sigma)^{q^{n-i}} ,$$

where the indices are taken modulo $n$. 

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Proposition 3.10. Because of cardinality reasons the condition $AD - BC \neq 0$ is necessary. Then
$$\left\{ \left( \frac{x}{f(x)} \right) : x \in \mathbb{F}_{q^n} \right\} = \left\{ \frac{x^\sigma}{f(x)^\sigma} : x \in \mathbb{F}_{q^n} \right\}$$
holds if and only if
$$C x^\sigma + D \sum_{j=0}^{n-1} a_j^2 x^{\sigma q^j} = \sum_{i=0}^{n-1} o_i^q \left( A x^\sigma + B \sum_{j=0}^{n-1} a_j^2 x^{\sigma q^j} \right)^q$$
for each $x \in \mathbb{F}_{q^n}$. After reducing modulo $x^{\sigma q^n} - x$, this is a polynomial equation of degree at most $q^{n-1}$ in the variable $x^\sigma$. It follows that it holds for each $x \in \mathbb{F}_{q^n}$ if and only if it is the zero polynomial. Comparing coefficients on both sides yields the assertion. 

We are able to prove the following.

Proposition 3.10. Consider a polynomial of the form $f(x) = \delta x^q + x^{q^{n-1}}$, where $q > 4$ is a power of the prime $p$. If $n > 4$, then for each generator $\delta$ of the multiplicative group of $\mathbb{F}_{q^n}$ the linear set $L_f$ is not simple.

Proof. Lemma 3.1 yields $L_f = L_{\delta}$ thus it is enough to show the existence of $\delta$ such that there is no $\mathbb{F}_{q^n}$-semilinear map between $U_f$ and $U_{\delta}$. In the equations of Proposition 3.9 we have $a_1 = \delta$, $a_{n-1} = 1$ and $a_0 = a_2 = \ldots = a_{n-2} = 0$. If $n > 4$ then the first two and the last two equations of Proposition 3.9 give

$$C = (B \delta^{q+1})^{q^{n-1}} + B^q,$$
$$D \delta^q - A^q = 0,$$
$$0 = (B \delta)^{q^{n-1}},$$
$$D - (\delta A)^{q^{n-1}} = 0,$$

where $\sigma = p^k$ for some integer $k$. If there is a solution, then $B = C = 0$ and $(\delta A)^{q^{n-1}} \delta^q = A^q$. Taking $q$-th powers on both sides yield

$$\delta^{q^{n+1}} = A^{q^2-1}$$

and hence

$$\delta^{(q^{n+1})(q^{n-1})^{-1}} = 1.$$
For each $\sigma$ let $G_\sigma$ be the set of elements $\delta$ of $\mathbb{F}_{q^r}$ satisfying (16). For each $\sigma$, $G_\sigma$ is a subgroup of the multiplicative group $M$ of $\mathbb{F}_{q^n}$. We show that these are proper subgroups of $M$. We have $G_{p^k} = M$ if and only if $q^n - 1$ divides $\frac{(p^k q^r + 1)(q^n - 1)}{q - 1}$, i.e. when $q - 1$ divides $p^k q + 1$. Since $\gcd(p^r q + 1, p^r - 1)$ is always 1, or $p^{\gcd(w,v)} + 1$, it follows that for $q > 4$ we cannot have $q - 1$ as a divisor of $p^k q + 1$.

It follows that for any generator $\delta$ of $M$ we have $\delta \notin \cup_j G_{p^j}$ and hence $\delta^{\sigma q + 1} \neq A^{q^2 - 1}$ for each $\sigma$ and for each $A$.

**Remark 3.11.** If $q = 4$, then (15) with $k = 2(n - 1) + 1$ asks for the solution of $\delta^3 = A^{15}$. When $n$ is odd, then $\{x^3 : x \in \mathbb{F}_{4^n}\} = \{x^{15} : x \in \mathbb{F}_{4^n}\}$ and hence for each $\delta$ there exists $A$ such that $\delta^3 = A^{15}$.

If $q = 3$, then (15) with $k = n - 1$ asks for the solution of $\delta^2 = A^8$. When $n$ is odd, then $\{x^2 : x \in \mathbb{F}_{3^n}\} = \{x^8 : x \in \mathbb{F}_{3^n}\}$ and hence for each $\delta$ there exists $A$ such that $\delta^2 = A^8$.

If $q = 2$, then (15) with $k = 0$ asks for the solution of $\delta^3 = A^3$. This equation always has a solution.

### 4 Linear sets of rank 4 of PG(1, $q^4$)

$\mathbb{F}_q$-linear sets of rank two of PG$(1, q^2)$ are the Baer sublines, which are equivalent. As we have mentioned in the introduction, subgeometries are simple linear sets, in fact they have $\mathcal{Z}(\Gamma \mathcal{L})$-class one (cf. [18, Theorem 2.6] and [14, Section 25.5]). There are two non-equivalent $\mathbb{F}_q$-linear sets of rank 3 of PG$(1, q^3)$, the linear sets of size $q^2 + q + 1$ and those of size $q^2 + 1$. Linear sets in both families are equivalent, since the stabilizer of a $q$-order subgeometry $\Sigma$ of $\Sigma^* = \text{PG}(2, q^4)$ is transitive on the set of those points of $\Sigma^* \setminus \Sigma$ which are incident with a line of $\Sigma$ and on the set of points of $\Sigma^*$ not incident with any line of $\Sigma$ (cf. Section 5.2 and [17]). In the first case we have the linear sets of pseudoregulus type with $\Gamma \mathcal{L}$-class 1 and $\mathcal{Z}(\Gamma \mathcal{L})$-class 2 (cf. Remark 5.6 and Example 5.1). In the second case we have the linear sets defined by Tr$_{q^4/q}$ with $\Gamma \mathcal{L}$-class and $\mathcal{Z}(\Gamma \mathcal{L})$-class 1 (cf. Theorem 3.7, see also [12, Corollary 6]).

From [6, Proposition 2.3] it follows that $\mathbb{F}_q$-linear sets of rank 5 in PG$(W, q^4) = \text{PG}(2, q^4)$ are simple. The orbits of 5-dimensional $\mathbb{F}_q$-subspaces of $W$ under $\Gamma \mathcal{L}(3, q^4)$ are also determined (cf. [6, pg. 54]). The results related to Rédéi type blocking sets allow to determine all the orbits of 4-dimensional $\mathbb{F}_q$-subspaces of a two-dimensional $\mathbb{F}_q$-space under the group $\Gamma \mathcal{L}(2, q^4)$. The aim of this section is to prove that $\mathbb{F}_q$-linear sets of rank 4 in
PG(1, q^4), with maximum field of linearity F_q, are simple (cf. Theorem 4.5), since this does not follow from the above mentioned simplicity of F_q-linear blocking sets. As a corollary, a list of orbits under PΓL(2, q^4) of F_q-linear sets of rank 4 in PG(1, q^4) can be deduced from [6, pg. 54].

4.1 Subspaces defining the same linear set

**Lemma 4.1.** Let \( f(x) = \sum_{i=0}^{3} a_i x^q^i \) and \( g(x) = \sum_{i=0}^{3} b_i x^q^i \) be two \( q \)-polynomials over \( F_q^4 \), such that \( L_f = L_g \). Then

\[
N(a_1) + N(a_2) + N(a_3) + a_1^{1+q^2} a_2^{q+q^3} + a_2^{q+q^3} a_3^{1+q^2} + \text{Tr} q^4 / q \left( a_1 a_2^{q+q^3} a_3^{3} \right) =
\]

\[
N(b_1) + N(b_2) + N(b_3) + b_1^{1+q^2} b_2^{q+q^3} + b_2^{q+q^3} b_3^{1+q^2} + \text{Tr} q^4 / q \left( b_1 b_2^{q+q^3} b_3^{3} \right).
\]

**Proof.** We are going to follow the proof of Lemma 3.6. As in that proof, we may assume \( a_0 = b_0 = 0 \). In Lemma 3.4 take \( d = 1 + q + q^2 + q^3 \). We obtain

\[
\sum_{1 \leq i,j,k,m \leq 3} a_i a_j a_k a_m \sum_{x \in F_q^4} x^{q^i+q^j+q^k+q^l} =
\]

\[
\sum_{1 \leq i,j,k,m \leq 3} b_i b_j b_k b_m \sum_{x \in F_q^4} x^{q^i+q^j+q^k+q^l} = 1 \text{ if and only if }
\]

\[
q^i+q^j+q^{k+2}+q^{m+3} \equiv q^i+q^{j+1}+q^{k+2}+q^{m-1} \equiv 1+q+q^2+q^3 \pmod{q^4-1},
\]

and zero otherwise. Suppose that the former case holds.

First consider \( k = 1 \). Then \( q^i+q^{j+1}+q^{k+2}+q^{m-1} \leq q^3+q^4+q^2+1 \) hence one of the following holds.

- If \( q^i+q^{j+1}+q^{k+2}+q^{m-1} = 1 + q + q^2 + q^3 \), then \( m = i = j = k = 1 \).
- If \( q^i+q^{j+1}+q^{k+2}+q^{m-1} = 1 + q + q^2 + q^3 + q^4 - 1 = q + q^2 + q^3 + q^4 \),

then \( \{i,j+1,k+2,m-1\} = \{1,2,3,4\} \), hence one of the following holds

\[
i = 1, j = 3, k = 1, m = 3,
\]

\[
i = 2, j = 3, k = 1, m = 2.
\]

Now consider the case \( k \geq 2 \). Then \( q^i+q^{j+1}+q^{k+2}+q^{m-1} \equiv q^i+q^{j+1}+q^{k-2}+q^{m-1} \leq q^4+q^2+q^3+2(q^4-1) \) hence one of the following holds.
Proposition 4.2. Let $f(x)$ and $g(x)$ be two $q$-polynomials over $\mathbb{F}_q$, such that $L_f = L_g$. If the maximum field of linearity of $f$ is $\mathbb{F}_q$, then

$$g(x) = f(\lambda x)/\lambda,$$

or

$$g(x) = \hat{f}(\lambda x)/\lambda.$$

Proof. By Proposition 2.3, the maximum field of linearity of $g$ is also $\mathbb{F}_q$. First note that $L_g = L_f$ when $g$ is as in the assertion (cf. Lemmas 3.1 and 3.2). Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$. Let $b_0 = a_0$. From (8) with $n = 4$ and $k = 2$ we have $a_1 a_3^q = b_1 b_3^q$ and $a_2^{q+1} + q^2 = b_2^{q+1} + q^2$, respectively. From (9) with $n = 4$ and $k = 2$ we obtain

$$a_1^{q+1} a_2^q + a_2 a_3^{q+q^2} = b_1^{q+1} b_2^q + b_2 b_3^{q+q^2}. \quad (17)$$

Note that $a_1 a_3^q = b_1 b_3^q$ implies

$$N(b_1) N(b_3) = N(a_1) N(a_3). \quad (18)$$

Multiplying (17) by $b_2$ and applying $a_2^{q+1} + q^2 = b_2^{q+1} + q^2$ yields:

$$b_2^2 b_3^{q+q} - b_2(a_1^{q+1} a_2^q + a_2 a_3^{q+q^2}) + b_1^{q+1} a_2^{q+1} = 0. \quad (19)$$
First suppose \( b_1 b_2 b_3 \neq 0 \). Then (19) is a second degree polynomial in \( b_2 \).

Applying \( a_1 a_3^q = b_1 b_3^q \) it is easy to see that the roots of (19) are

\[
b_{2,1} = \frac{a_1^{q+1} a_2^2}{b_3^q + q},
\]

\[
b_{2,2} = \frac{a_2 a_3^{q^2+q}}{b_3^q + q}.
\]

First we consider \( b_2 = b_{2,1} \). Then \( a_2^{1+q^2} = b_2^{1+q^2} \) yields \( N(a_2) = N(b_3) \) and hence \( N(b_1) = N(a_3) \). In particular, \( N(b_1/a_3^q) = 1 \) and hence \( b_1 = a_3 \lambda q \) for some \( \lambda \in \mathbb{F}_{q^4}^* \). From \( a_1 a_3^q = b_1 b_3^q \) we obtain \( b_3 = a_1^{q^3} a_3/b_1^{q^3} = a_1^{q^3} \lambda q^{q-1} \). Applying this we get \( b_2 = a_1^{q^3+1} a_2^2/ b_3^{q^2+q} = a_2^{q^2} \lambda q^{-1} \) and hence

\[
g(x) = a_0 x + a_1^{q^3-1} x^q + a_2^{q^2} \lambda q^{q-1} x^{q^2} + a_1^{q^3} \lambda q^{q-1} x^{q^3} = f(x)/\lambda.
\]

as we claimed.

Now consider \( b_2 = b_{2,2} \). Then \( a_2^{1+q^2} = b_2^{1+q^2} \) yields \( N(a_3) = N(b_3) \) and hence \( N(a_1) = N(b_1) \). Hence \( b_1 = a_1 \lambda q^{-1} \) for some \( \lambda \in \mathbb{F}_{q^4}^* \). From \( a_1 a_3^q = b_1 b_3^q \) we obtain \( b_3 = a_1^{q^3} a_3/b_1^{q^3} = a_3 \lambda q^{q-1} \). Applying this we obtain

\[
b_2 = a_2 a_3^{q^2+q}/b_3^{q^2+q} = a_2 \lambda q^{-1} \] and hence

\[
g(x) = a_0 x + a_1 \lambda q^{-1} x^q + a_2^{q^2} \lambda q^{q-1} x^{q^2} + a_1^{q^3} \lambda q^{q-1} x^{q^3} = f(x)/\lambda.
\]

If \( b_1 = b_3 = 0 \), then either \( b_2 = 0 \) and the maximum field of linearity of \( g(x) \) is \( \mathbb{F}_{q^4} \), or \( b_2 \neq 0 \) and the maximum field of linearity of \( g(x) \) is \( \mathbb{F}_{q^4}^* \). Thus we may assume \( b_1 \neq 0 \) or \( b_3 \neq 0 \).

First assume \( b_2 \neq 0 \) and \( b_1 = 0 \). Then \( b_3 \neq 0 \) and (19) gives

\[
b_2 a_3^{q^2+q} = a_1^{q+1} a_2^2 + a_2 a_3^{q^2+q}.
\]

Then \( a_1 a_3^q = b_1 b_3^q \) yields either \( a_1 = 0 \) and \( b_2 a_3^{q^2+q} = a_2 a_3^{q^2+q} \), or \( a_3 = 0 \) and \( b_2 a_3^{q^2+q} = a_1^{q+1} a_2^q \). Taking \( (q^2 + 1) \)-powers on both sides gives \( b_2^{q^2+1} N(b_3) = a_2^{q^2+1} N(a_3) \), or \( b_2^{q^2+1} N(b_1) = N(a_1) a_2^{q^2+1} \), respectively. Applying \( b_2^{q^2+1} = a_2^{q^2+1} \) we get \( N(b_1) = N(a_3) \), or \( N(b_3) = N(a_1) \), respectively. Note that the set of elements with norm 1 in \( \mathbb{F}_{q^4}^* \) is \( \{x^{q^2-1}; x \in \mathbb{F}_{q^4}^*\} \), thus in the first case there exists \( \lambda \in \mathbb{F}_{q^4}^* \) such that \( b_3 = a_3 \lambda q^{q-1} \). Then \( b_2 a_3^{q^2+q} = a_2 a_3^{q^2+q} \) yields \( b_2 = a_2 \lambda q^{q-1} \) and hence \( g(x) = a_0 x + a_2 \lambda q^{q-1} x^{q^2} + a_3 \lambda q^{q-1} x^{q^3} \).
In the second case the same reasoning yields $g(x) = a_0 x + a_2^q \lambda q^2 - 1 x^q + a_3^q \lambda q^3 - 1 x^q \lambda q^3$.

If $b_2 \neq 0$ and $b_3 = 0$, then the coefficient of $x^q$ in $\hat{g}(x)$ is zero and the assertion follows from the above arguments applied to $\hat{g}$ instead of $g$.

Now assume $b_2 = 0$ and $b_1 b_3 = 0$. Then $L_q = L_f$ is a linear set of pseudoregulus type and hence the assertion also follows from [16]. For the sake of completeness we present a proof also in this case. Equation $b_2^q + 1 = a_2^q + 1$ yields $a_2 = 0$ and equation $a_1 a_3^q = b_1 b_3^q$ yields $a_1 a_3 = 0$. Then from Lemma 4.1 we have

$$N(a_1) + N(a_3) = N(b_1) + N(b_3).$$

If $b_1 = 0$, then $b_3 \neq 0$ and either $a_1 = 0$ and $N(a_3) = N(b_3)$, or $a_3 = 0$ and $N(a_1) = N(b_3)$. In the first case $g(x) = a_0 x + a_3 \lambda q^3 - 1 x^q \lambda q^3$, in the second case $g(x) = a_0 x + a_3 \lambda q^3 - 1 x^q$. If $b_3 = 0$, then $b_1 \neq 0$ and either $a_1 = 0$ and $N(a_3) = N(b_1)$, or $a_3 = 0$ and $N(a_1) = N(b_1)$. In the first case $g(x) = a_0 x + a_2^q \lambda q^2 - 1 x^q \lambda q^2$, in the second case $g(x) = a_0 x + a_1 \lambda q^3 - 1 x^q$.

There is only one case left, when $b_2 = 0$ and $b_1 b_3 \neq 0$. Then from Lemma 4.1 and from $a_1 a_3^q = b_1 b_3^q$ it follows that

$$N(a_1) + N(a_3) = N(b_1) + N(b_3).$$

Together with (18) it follows that either $N(a_1) = N(b_1)$ and $N(a_3) = N(b_3)$, or $N(a_1) = N(b_3)$ and $N(a_3) = N(b_1)$. In the first case $g(x) = a_0 x + a_1 \lambda q^3 - 1 x^q \lambda q^2 + a_3 \lambda q^3 - 1 x^q \lambda q^3$, in the second case $g(x) = a_0 x + a_2^q \lambda q^2 - 1 x^q \lambda q^3 + a_1^q \lambda q^3 - 1 x^q \lambda q^3$, for some $\lambda \in \mathbb{F}_{q^4}$.

Now we are able to prove the following.

**Theorem 4.3.** Let $L_U$ be an $\mathbb{F}_q$-linear set of a line PG$(W, \mathbb{F}_{q^4})$ of rank 4, with maximum field of linearity $\mathbb{F}_q$, and let $\beta$ be any non-degenerate alternating form of $W$ over $\mathbb{F}_{q^4}$. If $V$ is an $\mathbb{F}_q$-vector subspace of $W$ such that $L_U = L_V$, then either

$$V = \mu U,$$

or

$$V = \mu U^\perp,$$

for some $\mu \in \mathbb{F}_{q^4}^*$, where $\perp$ is the orthogonal complement map induced by $\text{Tr}_{q^4/q} \circ \beta$ on the lattice of the $\mathbb{F}_q$-subspaces of $W$. 

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Proof. Assume that $L_U = L_V$ and let $\phi$ be a collineation of $\text{PG}(W, \mathbb{F}_{q^4})$ such that $L_U^\phi = L_V^\phi$ does not contain the point $((0, 1))_{\mathbb{F}_{q^4}}$. Denote by $\varphi$ the $\mathbb{F}_{q^4}$-semilinear map inducing the collineation $\phi$ and by $\sigma$ the associated field automorphism. Then $U^\varphi = U_f$ and $V^\varphi = V_g$ for some $q$-polynomials $f$ and $g$ over $\mathbb{F}_{q^4}$. By Proposition 4.2, taking also (6) into account, it follows that there exists $\lambda \in \mathbb{F}_{q^4}^*$ such that either $\lambda V^\varphi = U^\varphi$ or $\lambda V^\varphi = U^\varphi \perp'$, where $\perp'$ is the orthogonal complement map induced by the non-degenerate alternating form $\eta' = \text{Tr}_{q^4/q} \circ \eta$, with $\eta$ defined in (4). In the first case we have that $V = \mu U$, where $\mu = \frac{1}{\lambda^\varphi}$. In the second case we have $V = \frac{1}{\lambda^\varphi} U^\varphi \perp' \varphi^{-1}$. The map $\varphi \perp' \varphi^{-1}$ defines the orthogonal complement map on the lattice of the $\mathbb{F}_q$-subspaces of $W$ induced by the non-degenerate alternating form $\gamma': (u, v) \in W \times W \mapsto \text{Tr}_{q^n/q} \circ \gamma$, where

$$
\gamma(u, v) := \eta(\varphi(u), \varphi(v)).
$$

Since $\beta$ and $\gamma$ are two non-degenerate alternating forms of the 2-dimensional $\mathbb{F}_{q^4}$-space $W$, it follows that there exists $a \in \mathbb{F}_{q^4}^*$ such that $\beta(x, y) = a \gamma(x, y)$ for each $x, y \in W$. Hence, straightforward computation show that $U^\varphi \perp' \varphi^{-1} = a U^\perp \delta$. The assertion follows with $\mu = \frac{2}{\lambda^\varphi}$.

4.2 Semilinear maps between $U_f$ and $U_g$

The next result is just Proposition 3.9 with $n = 4$.

**Corollary 4.4.** Let $f(x) = a_0 x + a_1 x^q + a_2 x^{q^2} + a_3 x^{q^3}$. There is an $\mathbb{F}_{q^4}$-semilinear map between $U_f$ and $U_g$ if and only if the following system of four equations has a solution $A, B, C, D \in \mathbb{F}_{q^4}$, $AD - BC \neq 0$, $\sigma = q^k$.

\[
C + Da_0^\sigma - a_0 A = Ba_0 a_0^\gamma + (Ba_1 a_0^\gamma)^q + (Ba_2 a_0^\gamma)^{q^2} + (Ba_3 a_0^\gamma)^{q^3},
\]

\[
Da_1^\gamma - (a_3 A)g = Ba_0 a_1^\gamma + (Ba_1 a_1^\gamma)^q + (Ba_2 a_1^\gamma)^{q^2} + (Ba_3 a_1^\gamma)^{q^3},
\]

\[
Da_2^\gamma - (a_2 A)^q = Ba_0 a_2^\gamma + (Ba_1 a_2^\gamma)^q + (Ba_2 a_2^\gamma)^{q^2} + (Ba_3 a_2^\gamma)^{q^3},
\]

\[
Da_3^\gamma - (a_1 A)^{q^2} = Ba_0 a_3^\gamma + (Ba_1 a_3^\gamma)^{q^2} + (Ba_2 a_3^\gamma)^{q^3} + (Ba_3 a_3^\gamma)^q.
\]

**Theorem 4.5.** Linear sets of rank 4 of $\text{PG}(1, q^4)$, with maximum field of linearity $\mathbb{F}_q$, are simple.

**Proof.** Let $f = \sum_{i=0}^3 a_i x^{q^i}$. After a suitable projectivity we may assume $a_0 = 0$. We will use Corollary 4.4 with $\sigma \in \{1, q^2\}$. We may assume that
\(a_1 = 0\) and \(a_3 = 0\) do not hold at the same time since otherwise \(f\) is \(\mathbb{F}_{q^2}\)-linear.

First consider the case when \(N(a_1) = N(a_3)\). Let \(B = C = 0, D = A^{q^2}\) and take \(A\) such that \(A^{q-1} = a_3/a_1^q\). This can be done since \(N(a_3/a_1^q) = 1\). Then Corollary 4.4 with \(\sigma = q^2\) provides the existence of an \(\mathbb{F}_{q^4}\)-semilinear map between \(U_f\) and \(U_j\).

From now on we assume \(N(a_1) \neq N(a_3)\).

If \(a_2 = a_1 = 0\), then let \(\sigma = 1, A = D = 0, B = 1\) and \(C = a_3^{2q}\). If \(a_2 = a_3 = 0\), then let \(\sigma = 1, A = D = 0, B = 1\) and \(C = a_1^{2q}\).

Now consider the case \(a_2 \neq 0\) and \(a_1 a_3 \neq 0\). Let \(A = D = 0\). Then the equations of Corollary 4.4 with \(\sigma = 1\) yield

\[
C = B^{q^3} a_1^{2q^3} + B^{q^3} a_3^{2q^3}, \tag{22}
\]

\[
0 = B^q a_1^q a_3^q + B^q a_1^q a_3^q a_3^q. \tag{23}
\]

(23) is equivalent to \(0 = (Ba_1 a_3)^q + Ba_1 a_3\). Since \(X^{q^2} + X = 0\) has \(q^2\) solutions in \(\mathbb{F}_{q^4}\), for any \(a_1\) and \(a_3\) we can find \(B \in \mathbb{F}_{q^4}\) such that (23) is satisfied. If \(B^{q^3} a_1^{2q^3} + B^{q^3} a_3^{2q^3} \neq 0\), then let \(C\) be this field element. We show that this is always the case. Suppose, contrary to our claim, that \(B^{q^3} a_1^{2q^3} + B^{q^3} a_3^{2q^3} = 0\). Because of the choice of \(B\) (23) yields \(B^{q^3} a_1^{q^3} = -a_3^{q^3} a_3^{q^3}\). Since \(B \neq 0\) this implies

\[
-a_3^{2q^3}/a_1^{2q^3} = -a_3^{q^3} a_3^{q^3},
\]

and hence \(a_1^{q^2 + 1} = a_3^{q^2 + 1}\). A contradiction since \(N(a_1) \neq N(a_3)\). From now on we assume \(a_2 \neq 0\), we may also assume \(a_2 = 1\) after a suitable projectivity.

Corollary 4.4 with \(\sigma = 1\) yields

\[
C = (Ba_1^2)^{q^3} + B^{q^2} + (Ba_3^2)^q, \tag{24}
\]

\[
Da_1 - (a_3 A)^q = (Ba_1)^{q^3} + (Ba_3)^q, \tag{25}
\]

\[
D - A^{q^2} = (Ba_1 a_3)^{q^3} + (Ba_3 a_1)^q, \tag{26}
\]

\[
Da_3 - (a_1 A)^q = (Ba_1)^2 + (Ba_3)^q. \tag{27}
\]

The right hand side of (25) is the \(q\)-th power of the right hand side of (27) and hence \(D^q a_3^q - a_1 A = Da_1 - a_3^q A^q\), i.e.

\[
a_3^q(D + A)^q = a_1(D + A).
\]
Since $a_1$ or $a_3$ is non-zero, we have either $D = -A$, or $(D + A)^{q-1} = a_1/a_3^2$. The latter case can be excluded since in that case $N(a_1) = N(a_3)$. Let $D = -A$. Then the left hand side of (25) is $w(A) := -Aa_1 - a_3^2 A^q$. The kernel of $w$ is trivial and hence $B$ uniquely determines $A$. The inverse of $w$ is

$$w^{-1}(x) = \frac{-xa_1^{q^2+q^3} + x^3a_1^{q^2+q^3} a_1^{q^2+q^3} - x^3a_1^{q^2+q^3} a_1^{q^2+q^3} + x^3a_1^{q^2+q^3} a_1^{q^2+q^3}}{N(a_1) - N(a_3)}.$$

Denote the right hand side of (25) by $r(B)$, the right hand side of (26) by $t(B)$. Then $B$ has to be in the kernel of

$$K(x) := w^{-1}(r(x)) + (w^{-1}(r(x)))^q + t(x).$$

If $B = 0$, then $A = B = D = 0$ and hence this is not a suitable solution. It is easy to see that $Im t \subseteq \mathbb{F}_{q^2}$ and hence also $Im K \subseteq \mathbb{F}_{q^2}$, so the kernel of $K$ has at least dimension 2.

Let $B \in \ker K$, $B \neq 0$. Let $A := w^{-1}(r(B))$ and $C := (Ba_1^{q^2})^q + B^{q^2} + (Ba_3^{q^2})^q$ (we recall $D = -A$). This gives a solution. We have to check that $B$ can be chosen such that $AD - BC \neq 0$, i.e.

$$Q(B) := (w^{-1}(r(B)))^2 + B ( (Ba_1^{q^2})^q + B^{q^2} + (Ba_3^{q^2})^q ),$$

is non-zero. We have $w^{-1}(r(x))(N(a_1) - N(a_3)) = \sum_{i=0}^{3} c_i x^{q^i}$, where

$$c_0 = a_1^{1+q^2+q^3} a_3^q - a_1^{q^2+q^3} a_3^{1+q+q^2},$$

$$c_1 = a_3^{2q+q^2+q^3} - a_1^{q+q^3} a_3^{q+q^2},$$

$$c_2 = a_3^{q^2+q^3} a_1^q - a_1^{q+q^3} a_3^{q^2},$$

$$c_3 = a_3^{q^2+q^3} a_3^{q+q^3} - a_1^{q+q^2+2q^3}.$$
It follows that $Q$ is always singular and it has rank 2 or 3. In particular, the rank of $Q$ is 2 when the intersection of the planes $A : X_0 = 0$ and $B : X_1a_3^2 + X_2 + X_3a_1^2 = 0$ is contained in the plane $C : \sum_{i=0}^{3} c_i X_i = 0$. Straightforward calculations show that under our hypothesis ($a_1 \neq 0$ or $a_3 \neq 0, N(a_1) \neq N(a_3)$) this happens if and only if $1 = a_1^2 a_3$. We recall that the kernel of $K$ has dimension at least two. Let

$$H = \{ (x, x^q, x^{q^2}, x^{q^3})_{q^4} : K(x) = 0 \}.$$ 

Our aim is to prove that $H$ has points not belonging to the quadric $Q$, i.e. $H \not\subseteq Q$.

Note that $x \in F_{q^4} \mapsto (x, x^q, x^{q^2}, x^{q^3}) \in F_{q^4}$ is a vector-space isomorphism between $F_{q^4}$ and the 4-dimensional $F_q$-space $\{(x, x^q, x^{q^2}, x^{q^3}) : x \in F_{q^4} \} \subset F_{q^4}$. Denote by $\bar{H}$ the $F_{q^4}$-extension of $H$, i.e. the projective subspace of $PG(3, q^4)$ generated by the points of $H$. Then the projective dimension of $\bar{H}$ is $\dim \ker K - 1$. Let $\xi$ denotes the collineation $(X_0, X_1, X_2, X_3) \mapsto (X_0^q, X_1^q, X_2^q, X_3^q)$ of $PG(3, q^4)$. Then the points of $\bar{H}$ are fixed points of $\xi$ and hence $\xi$ fixes the subspace $\bar{H}$. Note that the subspace of singular points of $Q$ is always disjoint from $H$ since it is contained in $A$, while $H$ is disjoint from it.

First of all note that if $\dim \ker K = 4$, i.e. $K$ is the zero polynomial, then $H$ is a subgeometry of $PG(3, q^4)$ isomorphic to $PG(3, q)$, which clearly cannot be contained in $Q$. It follows that $\dim \ker K$ is either 3 or 2, i.e. $H$ is either a $q$-order subplane or a $q$-order subline.

First assume $1 \neq a_1^2 a_3$, i.e. the case when $Q$ has rank 3. If $H$ is a $q$-order subplane, then $H$ cannot be contained in $Q$. To see this, suppose the contrary and take three non-concurrent $q$-order sublines of $H$. The $F_{q^4}$-extensions of these sublines are also contained in $Q$, but there is at least one of them which does not pass through the singular point of $Q$, a contradiction. Now assume that $H$ is a $q$-order subline. The singular point of $Q$ is the intersection of the planes $A, B$ and $C$. Straightforward calculations show that this point is $V = \langle (v_0, v_1, v_2, v_3) \rangle_{q^4}$, where

$$v_0 = 0,$$

$$v_1 = a_1^{q^2+q^3} (a_1^{q^3} - a_3^2 - 1),$$

$$v_2 = a_1^q a_3^q (a_1^{q^2} a_3^q - a_1^{q^3} a_3^2),$$

$$v_3 = a_3^{q+q^2} (1 - a_1^{q^2} a_3^q).$$

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Suppose, contrary to our claim, that $H$ is contained in $Q$. Then $\tilde{H}$ passes through the singular point $V$ of $Q$. Since $\tilde{H}$ is fixed by $\xi$, it follows that the points $V, V^{1}, V^{2}, V^{3}$ have to be collinear ($v_{0} = 0$ yields that these four points cannot coincide). Let $M$ denote the $4 \times 4$ matrix, whose $i$-th row consists of the coordinates of $V^{i-1}$ for $i = 1, 2, 3, 4$. The rank of $M$ is two, thus each of its minors of order three is zero. Let $M_{i,j}$ denote the submatrix of $M$ obtained by deleting the $i$-th row and $j$-th column of $M$. Then
\[
\det M_{1,2} = a_{1}^{q+1}(a_{1}^{q}a_{3} - 1)^{q+1}\alpha,
\]
\[
\det M_{1,4} = a_{3}^{q+1}(a_{3}^{q}a_{3} - 1)^{q+1}\beta,
\]
where
\[
\alpha = N(a_{1})(a_{1}^{q}a_{3}^{q} - 1) + N(a_{3})(1 - a_{1}^{q}a_{3} - a_{1}^{q}a_{3}^{q} + a_{1}a_{3}^{q}),
\]
\[
\beta = N(a_{1})(a_{1}^{q}a_{3}^{q} + a_{2}^{q}a_{3}^{q} - a_{1}^{q}a_{3} - 1) + N(a_{3})(1 - a_{1}^{q}a_{3}^{q}).
\]
Since $a_{1}$ and $a_{3}$ cannot be both zeros and $a_{1}^{q}a_{3} - 1 \neq 0$, we have $\alpha = \beta = 0$. But $\alpha - \beta = (N(a_{1}) - N(a_{3}))(a_{1}^{q}a_{3} - a_{1}a_{3}^{q})$. It follows that $a_{1}^{q}a_{3} \in \mathbb{F}_{q}$ and hence $\alpha$ can be written as $(N(a_{1}) - N(a_{3}))(a_{1}^{q}a_{3} - 1)$, which is non-zero.

This contradiction shows that $V$ cannot be contained in a line fixed by $\xi$ and hence $\tilde{H}$ cannot pass through $V$. It follows that $H \not\subset Q$ and hence we can choose $B$ such that $AD - BC \neq 0$.

Now consider the case $1 = a_{1}^{q}a_{3}$. Then $Q$ is the union of two planes meeting each other in $\ell := A \cap B$. It is easy to see that $R := \langle (0, 1, -a_{3}^{2}, 0) \rangle_{q^{2}}$ and $R^{3}$ are two distinct points of $\ell$. Since $N(a_{1}) \neq N(a_{3})$ and $N(a_{1})N(a_{3}) = 1$, $\det \{R, R^{3}, R^{2}, R^{3} \} = N(a_{3})^{2} - 1$ cannot be zero and hence $R \not\subset H$, otherwise $\dim \langle R, R^{3}, R^{2}, R^{3} \rangle = \dim \tilde{H} \leq 2$. Suppose, contrary to our claim, that $H$ is contained in one of the two planes of $Q$. Since $R \not\subset H$, such a plane can be written as $\langle H, R \rangle$ and since $H$ is fixed by $\xi$ and $\ell \subseteq \langle H, R \rangle$, we have $\langle H, R \rangle^{3} = \langle H, R^{3} \rangle = \langle H, R \rangle$. Thus $R, R^{3}, R^{2}$, $R^{3}$ are coplanar, a contradiction.

\[\square\]

5 Different aspects of the classes of a linear set

5.1 Class of a linear set and the associated variety

Let $L_{W}$ be an $\mathbb{F}_{q}$-linear set of rank $k$ of $\text{PG}(W, \mathbb{F}_{q^{n}}) = \text{PG}(r-1, q^{n})$. Consider the projective space $\Omega = \text{PG}(W, \mathbb{F}_{q}) = \text{PG}(rn - 1, q)$. For each point $P =$
\( \langle u \rangle_{F_q^n} \) of \( \text{PG}(W, F_q^n) \) there corresponds a projective \((n-1)\)-subspace \( X_P := \text{PG}(\langle u \rangle_{F_q^n}, F_q) \) of \( \Omega \). The variety of \( \Omega \) associated to \( L_U \) is

\[
V_{r,n,k}(L_U) = \bigcup_{P \in L_U} X_P. \tag{28}
\]

This variety was already used in [2] and [16], see Example 5.1. The question of determining whether a linear set is simple or not is related to the existence of so-called \textit{irregular subspaces} (see [16]). The case of irregular sublines was already studied in [11].

A \((k-1)\)-space \( \mathcal{H} = \text{PG}(V, F_q) \) of \( \Omega \) is said to be a \textit{transversal} space of \( V(L_U) \) if \( \mathcal{H} \cap X_P \neq \emptyset \) for each point \( P \in L_U \), i.e. \( L_U = L_V \).

The \( Z(\Gamma L) \)-class of an \( F_q \)-linear set \( L_U \) of rank \( n \) of \( \text{PG}(W, F_{q^n}) = \text{PG}(1, q^n) \), with maximum field of linearity \( F_q \), is the number of transversal spaces of \( V_{2,n,n}(L_U) \) up to the action of the subgroup \( G \) of \( \text{PGL}(2n-1, q) \) induced by the maps \( x \in W \mapsto \lambda x \in W \), with \( \lambda \in F_q^* \). Note that \( G \) fixes \( X_P \) for each point \( P \in \text{PG}(1, q^n) \) and hence fixes the variety.

The maximum size of an \( F_q \)-linear set \( L_U \) of rank \( n \) of \( \text{PG}(1, q^n) \) is \( (q^n - 1)/(q - 1) \). If this bound is attained (hence each point of \( L_U \) has weight one), then \( L_U \) is a \textit{maximum scattered} linear set of \( \text{PG}(1, q^n) \). For maximum scattered linear sets, the number of transversal spaces through \( Q \in V(L_U) \) does not depend on the choice of \( Q \) and this number is the \( Z(\Gamma L) \)-class of \( L_U \).

**Example 5.1.** Let \( U = \{ (x, x^q) : x \in F_q^n \} \) and consider the linear set \( L_U \). In [16] the variety \( V_{2,n,n}(L_U) \) was studied, and the transversal spaces were determined. It follows that the \( Z(\Gamma L) \)-class of \( L_U \) is \( \varphi(n) \), where \( \varphi \) is the Euler’s phi function.

### 5.2 Classes of linear sets as projections of subgeometries

Let \( \Sigma = \text{PG}(k-1, q) \) be a canonical subgeometry of \( \Sigma^* = \text{PG}(k-1, q^n) \). Let \( \Gamma \subset \Sigma^* \setminus \Sigma \) be a \((k-r-1)\)-space and let \( \Lambda \subset \Sigma^* \setminus \Gamma \) be an \((r-1)\)-space of \( \Sigma^* \). The projection of \( \Sigma \) from \textit{center} \( \Gamma \) to \textit{axis} \( \Lambda \) is the point set

\[
L = p_{\Gamma,\Lambda}(\Sigma) := \{ (\Gamma, P) \cap \Lambda : P \in \Sigma \}. \tag{29}
\]

In [23] Lunardon and Polverino characterized linear sets as projections of canonical subgeometries. They proved the following.

**Theorem 5.2** ([23, Theorems 1 and 2]). Let \( \Sigma^*, \Sigma, \Lambda, \Gamma \) and \( L = p_{\Gamma,\Lambda}(\Sigma) \) be defined as above. Then \( L \) is an \( F_q \)-linear set of rank \( k \) and \( \langle L \rangle = \Lambda \).
Conversely, if $L$ is an $\mathbb{F}_q$-linear set of rank $k$ of $\Lambda = \text{PG}(r - 1, q^n) \subset \Sigma^*$ and $(L) = \Lambda$, then there is a $(k - r - 1)$-space $\Gamma$ disjoint from $\Lambda$ and a canonical subgeometry $\Sigma = \text{PG}(r - 1, q)$ disjoint from $\Gamma$ such that $L = p_{\Gamma, \Lambda}(\Sigma)$.

Let $L_U$ be an $\mathbb{F}_q$-linear set of rank $k$ of $\Sigma = \text{PG}(W, \mathbb{F}_{q^n})$ such that for each $k$-dimensional $\mathbb{F}_q$-subspace $V$ of $W$ if $\text{PG}(V, \mathbb{F}_q)$ is a transversal space of $V_{r,n,k}(L_U)$, then there exists $\gamma \in \text{PGL}(W, \mathbb{F}_q)$, such that $\gamma$ fixes the Desarguesian spread $\{X_P : P \in \mathbb{P}\}$ and $\text{PG}(U, \mathbb{F}_q)^\gamma = \text{PG}(V, \mathbb{F}_q)$. This is condition (A) from [7], and it is equivalent to say that $L_U$ is a simple linear set. Then the main results of [7] can be formalized as follows.

Theorem 5.3 ([7]). Let $L_1 = p_{\Gamma_1, \Lambda_1}(\Sigma_1)$ and $L_2 = p_{\Gamma_2, \Lambda_2}(\Sigma_2)$ be two linear sets of rank $k$. If $L_1$ and $L_2$ are equivalent and one of them is simple, then there is a collineation mapping $\Gamma_1$ to $\Gamma_2$ and $\Sigma_1$ to $\Sigma_2$.

Theorem 5.4 ([7]). If $L$ is a non-simple linear set of rank $k$ in $\Lambda = (L)$, then there is a subspace $\Gamma = \Gamma_1 = \Gamma_2$ disjoint from $\Lambda$, and two $q$-order canonical subgeometries $\Sigma_1, \Sigma_2$ such that $L = p_{\Gamma, \Lambda}(\Sigma_1) = p_{\Gamma, \Lambda}(\Sigma_2)$, and there is no collineation fixing $\Gamma$ and mapping $\Sigma_1$ to $\Sigma_2$.

Now we interpret the classes of linear sets, hence we are going to consider $\mathbb{F}_q$-linear sets of rank $n$ of $\Lambda = (\Sigma)$, with maximum field of linearity $\mathbb{F}_q$. Arguing as in the proof of [7, Theorem 7], if $L_U$ is non-simple, then for any pair $U, V$ of $n$-dimensional $\mathbb{F}_q$-subspaces of $W$ with $L_U = L_V$ such that $U^f \neq V$ for each $f \in \Gamma L(2, q^n)$ we can find a $q$-order subgeometry $\Sigma$ of $\Sigma^* = \text{PG}(n - 1, q^n)$ and two $(n - 3)$-spaces $\Gamma_1$ and $\Gamma_2$ of $\Sigma^*$, disjoint from $\Sigma$ and from $\Lambda$, lying on different orbits of $\text{Stab}(\Sigma)$. On the other hand, arguing as in [7, Theorem 6], if there exist two $(n - 3)$-subspaces $\Gamma_1$ and $\Gamma_2$ of $\Sigma^*$, disjoint from $\Sigma$ and from $\Lambda$, belonging to different orbits of $\text{Stab}(\Sigma)$ and such that $L = p_{\Lambda, \Gamma_1}(\Sigma) = p_{\Lambda, \Gamma_2}(\Sigma)$, then it is possible to construct two $n$-dimensional $\mathbb{F}_q$-subspaces $U$ and $V$ of $W$ with $L_U = L_V$ such that $U^f \neq V$ for each $f \in \Gamma L(2, q^n)$. Hence we can state the following.

The $\Gamma L$-class of $L_U$ is the number of orbits of $\text{Stab}(\Sigma)$ on $(n - 3)$-spaces of $\Sigma^*$ containing a $\Gamma$ disjoint from $\Sigma$ and from $\Lambda$ such that $p_{\Lambda, \Gamma}(\Sigma)$ is equivalent to $L_U$.

5.3 Class of linear sets and linear blocking sets of Rédei type

A blocking set $B$ of $\text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(2, q^n)$ is a point set meeting every line of the plane. Blocking sets of size $q^n + N \leq 2q^n$ with an $N$-secant are called blocking sets of Rédei type, the $N$-secants of the blocking set are called
Rédei lines. Let $L_U$ be an $\mathbb{F}_q$-linear set of rank $n$ of a line $\ell = \text{PG}(W, \mathbb{F}_{q^n})$, $W \leq V$, and let $w \in V \setminus W$. Then $(U, w)_{\mathbb{F}_q}$ defines an $\mathbb{F}_q$-linear blocking set of $\text{PG}(2, q^n)$ with Rédei line $\ell$. The following theorem tells us the number of inequivalent blocking sets obtained in this way.

**Theorem 5.5.** The $\Gamma L$-class of an $\mathbb{F}_q$-linear set $L_U$ of rank $n$ of $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$, with maximum field of linearity $\mathbb{F}_q$, is the number of inequivalent $\mathbb{F}_q$-linear blocking sets of Rédei type of $\text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(2, q^n)$ containing $L_U$.

**Proof.** $\mathbb{F}_q$-linear blocking sets of $\text{PG}(2, q^n)$ with more than one Rédei line are equivalent to those defined by $\text{Tr}_{q^n/q}(x)$ for some divisor $m$ of $n$, see [21, Theorem 5]. Suppose first that $L_U$ is equivalent to $L_T$, where $T = \{(x, \text{Tr}_{q^n/q}(x)): x \in \mathbb{F}_{q^n}\}$. According to Theorem 3.7 $L_T$, and hence also $L_U$, have $\mathbb{Z}(\Gamma L)$-class and $\Gamma L$-class one and hence there exists a unique point $P \in L_U$ such that $w_{L_U}(P) = n - 1$. Then for each $v \in V \setminus W$ the $\mathbb{F}_q$-linear blocking set defined by $(U, v)_{\mathbb{F}_q}$ has more than one Rédei line, each of them incident with $P$, and hence it is equivalent to the Rédei type blocking set obtained from $\text{Tr}_{q^n/q}(x)$.

Now let $B_1 = L_{U_1}$ and $B_2 = L_{U_2}$ be two $\mathbb{F}_q$-linear blocking sets of Rédei type with $\text{PG}(W, \mathbb{F}_{q^n})$ the unique Rédei line. Denote by $U_1$ and $U_2$ the $\mathbb{F}_q$-subspaces $W \cap V_1$ and $W \cap V_2$, respectively, and suppose $L_{U_1} = L_{U_2}$ with $\mathbb{F}_q$ the maximum field of linearity. Then $B_1$ and $B_2$ have $(q + 1)$-secants and we have $V_1 = U_1 \oplus \langle u_1 \rangle_{\mathbb{F}_q}$ and $V_2 = U_2 \oplus \langle u_2 \rangle_{\mathbb{F}_q}$ for some $u_1, u_2 \in V \setminus W$.

If $B_1^\ell = B_2$, then [6, Proposition 2.3] implies $V_1^\ell = \lambda V_2$ for some $\lambda \in \mathbb{F}_{q^n}^*$. Such $f \in \Gamma L(3, q^n)$ has to fix $W$ and it is easy to see that $U_1^\ell = \lambda U_2$, i.e. $U_1$ and $U_2$ are $\Gamma L(2, q^n)$-equivalent.

Conversely, if there exists $f \in \Gamma L(W, \mathbb{F}_{q^n})$ such that $U_1^\ell = U_2$, then $B_1^{\ell g} = B_2$, where $g \in \Gamma L(V, \mathbb{F}_{q^n})$ is the extension of $f$ mapping $u_1$ to $u_2$. \qed

### 5.4 Class of linear sets and MRD-codes

In [27, Section 4] Sheekey showed that maximum scattered $\mathbb{F}_q$-linear sets of $\text{PG}(1, q^n)$ yield $\mathbb{F}_q$-linear maximum rank distance codes (MRD-codes) of dimension $2n$ and minimum distance $n - 1$, that is, a set $\mathcal{M}$ of $q^{2n} n \times n$ matrices over $\mathbb{F}_q$ forming an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^{2n \times n}$ of dimension $2n$ such that the non-zero matrices of $\mathcal{M}$ have rank at least $n - 1$. It can be easily seen that these MRD-codes have the so-called middle nucleus isomorphic to $\mathbb{F}_{q^n}$.

For definitions and properties on MRD-codes we refer the reader to [10] by Delsarte and [13] by Gabidulin. The kernel and the nuclei of MRD-codes are studied in [25].
For $n \times n$ matrices there are two different definitions of equivalence for MRD-codes in the literature. The arguments of [27, Section 4] yield the following interpretation of the $\Gamma L$-class:

- $\mathcal{M}$ and $\mathcal{M}'$ are equivalent if there are invertible matrices $A, B \in \mathbb{F}_q^{n \times n}$ and a field automorphism $\sigma$ of $\mathbb{F}_q$ such that $A\mathcal{M}'B = \mathcal{M}'$, see [27]. In this case the $\Gamma L$-class of $L_U$ is the number of inequivalent MRD-codes obtained from the linear set $L_U$.

- $\mathcal{M}$ and $\mathcal{M}'$ are equivalent if there are invertible matrices $A, B \in \mathbb{F}_q^{n \times n}$ and a field automorphism $\sigma$ of $\mathbb{F}_q$ such that $A\mathcal{M}'B = \mathcal{M}'$, or $A\mathcal{M}'^T B = \mathcal{M}'$, see [9]. In this case the number of inequivalent MRD-codes obtained from the linear set $L_U$ is between $\lfloor s/2 \rfloor$ and $s$, where $s$ is the $\Gamma L$-class of $L_U$.

We summarize here the known non-equivalent families of MRD-codes arising from maximum scattered linear sets.

1. $L_{U_1} := \{(x^s)_{\mathbb{F}_q^n} : x \in \mathbb{F}_q^*\}$ ([5]) gives Gabidulin codes,

2. $L_{U_2} := \{(x^s)_{\mathbb{F}_q^n} : x \in \mathbb{F}_q^*, \gcd(s, n) = 1\}$ ([5]) gives generalized Gabidulin codes,

3. $L_{U_3} := \{(x, \delta x^s + x^{q^n-1})_{\mathbb{F}_q^n} : x \in \mathbb{F}_q^*\}$ ([22]) gives MRD-codes found by Sheekey in [27],

4. $L_{U_4} := \{(x, \delta x^s + x^{q^n-1})_{\mathbb{F}_q^n} : x \in \mathbb{F}_q^*, \gcd(s, n) = 1\}$ gives MRD-codes found by Sheekey in [27] and studied by Lunardon, Trombetti and Zhou in [24].

**Remark 5.6.** The linear sets $L_{U_1}$ and $L_{U_2}$ coincide, but when $s \notin \{1, n-1\}$, there is no $f \in \Gamma L(2, q^n)$ such that $U_1^f = U_2$. These linear sets are of pseudoregulus type, [20] (see also Example 5.1), and in [7] it was proved that the $\Gamma L$-class of these linear sets is $\varphi(n)/2$, hence they are examples of non-simple linear sets for $n = 5$ and $n > 6$.

It can be proved that the family $L_{U_4}$ contains linear sets non-equivalent to those from the other families. We will report on this elsewhere.
References


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